Strongly Nonlinear Parabolic Unilateral Problems Without Sign Conditions and Three Unbounded Nonlinearities

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Abstract

In this paper, we prove the existence of solutions to unilateral problems involving nonlinear operators of the formwhere $b(x,u)$ is unbounded function on u and A is a Leray - Lions operator from $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$. The nonlinearity $H(x,t,u,Du)$ satisfies the following growth condition $H(x,t,s,\xi)\le\gamma(x,t)+g(s)\,|\xi|^p$ with $\gamma\in L^1(Q)$ and $g\in L^1(\mathbb{R})$, and without assuming the sign condition on H. The second term f belongs to $L^1(Q)$ and $b(x,u_0)\in L^1(\Omega)$.

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1. Introduction

Introduction

The objective of this paper is to study the obstacle problem with L^1 data associated to nonlinear operator of the form

$$\frac{\partial b(x,u)}{\partial t}+Au+H(x,t,u,Du)=f. \quad (1.1)$$

The principal part A is a differential parabolic operator of the second order in divergence form, acting from $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$, defined as:

$$Au=-\mathrm{div}(a(x,t,u,Du)),$$

where Ω is a bounded subset of $\mathbb{R}^N$, $N\ge 2$, and $b(x,u)$ is an unbounded term. H is a nonlinear lower order term satisfying the following growth condition

$$H(x,t,s,\xi)\le\gamma(x,t)+g(s)\,|\xi|^p$$

with $\gamma\in L^1(Q)$ and $g\in L^1(\mathbb{R})$. The data f and $b(x,u_0)$ are, respectively, in $L^1(Q)$ and $L^1(\Omega)$. More precisely, this paper deals with the existence of solutions to the following problem

$$\begin{cases}
u\ge\psi \quad a.e \quad in \quad \Omega\times(0,T).\\
T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega)),\\
b(x,u)\in L^\infty\left([0,T],L^1(\Omega)\right)\\
\displaystyle\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u|\le m+1\}}a(x,t,u,Du)Du\,dx\,dt=0,\\
\displaystyle -\int_\Omega B_S(x,u_0)v(x,0)dx-\int_0^T\int_\Omega\frac{\partial v}{\partial t}B_S(x,u)dxdt\\
\displaystyle +\int_Q S'(u)a(x,t,u,Du)Dvdxdt+\int_Q S''(u)a(x,t,u,Du)Duvdxdt\\
\displaystyle +\int_Q H(x,t,u,Du)S'(u)vdxdt\le\int_Q fS'(u)vdxdt,\\
\displaystyle \forall\ v\in K_\psi\cap L^\infty(Q),\ \frac{\partial v}{\partial t}\in L^{p'}(0,T;W^{-1,p'}(\Omega))\ with\ v(x,T)=0\\
such\ \ that\ \ S'(u)v\in L^p(0,T;W_0^{1,p}(\Omega)),\\
\forall\ \ S\in W^{2,\infty}(\mathbb{R})\ \ with\ \ S'\ \ has\ \ a\ \ compact\ \ support\ \ in\ \ \mathbb{R},\\
\displaystyle and\ B_S(x,r)=\int_0^r\frac{\partial b(x,\sigma)}{\partial\sigma}S'(\sigma)d\sigma,
\end{cases} \quad (1.2)$$

where ψ a measurable function with values in $\overline{\mathbb{R}}$ such that $\psi^+\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and

$$K_\psi = \left\{ u \in L^p(0,T; W_0^{1\ p}(\Omega)),\quad u \geq \psi \ \ a.e\ \ in\ \ \Omega \times (0,T) \right\},$$

Our principal goal in this paper is to prove the existence result for the unilateral parabolic problem (1.2) without assuming any sign condition on H and the term $b(x,u)$ is an unbounded function.

Porretta has proved in [15] the existence result for the problem (1.1) in the case of an elliptic equations with a measure right hand. Another result in this direction can be found i[2, 3, 6, 22] where the problem (1.1) is studied in elliptic case for inequality with $f \in L^1(\Omega)$.

For the parabolic equations we list the works of Landes [12] with $b(x,u) = u$ and $H = 0$ and $f \in L^{p'}(0,T; W^{-1,p'}(\Omega))$. A generalization of the last works in the case of $b(x,u) = u$ and $H \neq 0$ is treated in [13] (see also [8, 9, 10] for related topics). In the case of $f \in L^1(Q)$ and $b(x,u) = u$, see [17, 18].

In the case of $H(x,t,u,Du) = div(\phi(u))$ is studied by H. Redwane in the classical Sobolev spaces $W^{1,p}(\Omega)$ and Orlicz spaces , the assumptions for the parabolic part is inspired by [19, 20].

The plan of the paper is as follows: In Section 2 we make precise all the assumptions on $b,\ a,\ H,\ f\ \ and\ \ b(x,u_0)$, and the statement of result. In section 3 we prove our main result. In section 4 we give an appendix.

Assumptions on Data and statement of the result

Throughout the paper, we assume that the following assumptions hold true.

Assumption (H1) see[19,20].

Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$,

$$b : \Omega \times \mathbb{R} \to \mathbb{R} \qquad \text{is a Carathéodory function.}$$

(2.1.1)

such that for every $x \in \Omega$, $b(x,.)$ is a strictly increasing $C^1 -$ function with $b(x,0) = 0$.

Next, for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^1(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x)\ \ and\ \ \left| D_x \left(\frac{\partial b(x,s)}{\partial s} \right) \right| \leq B_k(x)$$

(2.1.2)

for almost every $x \in \Omega$, for every s such that $|s| \leq k$, we denote by $D_x\left(\frac{\partial b(x,s)}{\partial s}\right)$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions.

$$|a(x,t,s,\xi)| \leq \beta[k(x,t) + |s|^{p-1} + |\xi|^{p-1}],$$

(2.1.3)

for a.e. $(x,t) \in Q$,all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, some function $k(x,t) \in L^{p'}(Q)$ and $\beta > 0$.

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0 \quad \text{for all } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N,$$
$$a(x,t,s,\xi).\xi \geq \alpha|\xi|^p,$$

(2.1.4)

where α are strictly positive constant.

Assumption (H2).

Furthermore, let $H(x,t,s,\xi) :\ \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \ \to \mathbb{R}$ be a Carathéodory function such that for a.e $(x,t) \in Q$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s)|\xi|^p,$$

(2.2.1)

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x,t)$ belongs to $L^1(Q)$.

$$f \quad \text{is an element of} \quad L^1(Q),$$
$$b(x,u_0) \quad \text{is an element of} \quad L^1(\Omega).$$

(2.2.2)

Finally let ψ be a measurable function with values in $\overline{\mathbb{R}}$ such that

$$\psi^+ \in L^p(0,T; W_0^{1\ p}(\Omega)) \cap L^\infty(Q),$$

(2.2.3)

and let us define

$$K_\psi = \left\{ u \in L^p(0,T; W_0^{1\ p}(\Omega)),\quad u \geq \psi \ \ a.e\ \ in\ \ \Omega \times (0,T) \right\}.$$

(2.2.4)

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as,

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

The aim of this paper is to prove the following

Theorem 0.1 *Assume that the assumptions (??)- (??) hold. Then, the following problem:*

$$\begin{cases} u \geq \psi \quad a.e \quad in \quad \Omega \times (0,T). \\ T_k(u) \in L^p(0,T;W_0^{1\ p}(\Omega)), \quad b(x,u) \in L^\infty\left([0,T],L^1(\Omega)\right) \\ \lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Dudxdt = 0, \\ -\int_\Omega B_S(x,u_0)v(x,0)dx - \int_0^T\int_\Omega \frac{\partial v}{\partial t}B_S(x,u)dxdt \\ +\int_Q S'(u)a(x,t,u,Du)Dvdxdt + \int_Q S''(u)a(x,t,u,Du)Duvdxdt \\ +\int_Q H(x,t,u,Du)S'(u)vdxdt \leq \int_Q fS'(u)vdxdt, \\ \forall\ v \in K_\psi \cap L^\infty(Q),\ \frac{\partial v}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \ with \ v(x,T)=0 \\ such \ that \ \ S'(u)v \in L^p(0,T;W_0^{1\ p}(\Omega)), \\ \forall \ \ S \in W^{2,\infty}(\mathbb{R}) \ \ with \ \ S' \ \ has \ \ a \ \ compact \ \ support \ \ in \ \ \mathbb{R}, \\ B_S(x,r) = \int_0^r \frac{\partial b(x,\sigma)}{\partial\sigma}S'(\sigma)d\sigma,. \end{cases} \tag{1}$$

has at least one solution .

Remark 0.1 *Let us remark that in the case of $\psi = -\infty$ Theorem ?? states the existence of solution in the case of equation i.e. the following problem*

$$\begin{cases} T_k(u) \in L^p(0,T;W_0^{1\ p}(\Omega)), \quad b(x,u) \in L^\infty\left([0,T],L^1(\Omega)\right) \\ \lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Dudxdt = 0, \\ -\int_\Omega B_S(x,u_0)v(x,0)dx - \int_0^T\int_\Omega \frac{\partial v}{\partial t}B_S(x,u)dxdt \\ +\int_Q S'(u)a(x,t,u,Du)Dvdxdt + \int_Q S''(u)a(x,t,u,Du)Duvdxdt \\ +\int_Q H(x,t,u,Du)S'(u)vdxdt = \int_Q fS'(u)vdxdt, \\ \forall\ v \in L^p(0,T;W_0^{1\ p}(\Omega)) \cap L^\infty(Q),\ \frac{\partial v}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \ with \ v(x,T)=0 \\ such \ that \ \ S'(u)v \in L^p(0,T;W_0^{1\ p}(\Omega)), \\ \forall \ \ S \in W^{2,\infty}(\mathbb{R}) \ \ with \ \ S' \ \ has \ \ a \ \ compact \ \ support \ \ in \ \ \mathbb{R}, \end{cases} \tag{1}$$

has at least one solution.

Proof of Theorem 0.1

<u>Approximate problem</u>

For $n > 0$, let us define the following approximation of $b,\ H\ , f$ and u_0:

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}r \quad for \quad n > 0.$$

(3.1.1)

In view of (3.1.2), b_n is a Carathéodory function and satisfies (3.1.3), there exist $\lambda_n > 0$ and functions $A_n \in L^1(\Omega)$ and $B_n \in L^p(\Omega)$ such that

$$\lambda_n \leq \frac{\partial b_n(x,s)}{\partial s} \leq A_n(x) \ \ and \ \ \left|D_x\left(\frac{\partial b_n(x,s)}{\partial s}\right)\right| \leq B_n(x) \ \ a.e.in \ \Omega, \ \ s \in \mathbb{R}.$$

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}.$$

$$f_n \in L^{p'}(Q) \ \ and \ \ f_n \to f \ \ a.e \ in \ Q \ and \ strongly \ in \ L^1(Q) \ \ as \ \ n \to +\infty,$$

(3.1.4)

$$u_{0n} \in \mathcal{D}(\Omega): \quad \|b_n(x,u_{0n})\|_{L^1} \leq \|b(x,u_0)\|_{L^1},$$

$$b_n(x,u_{0n}) \to b(x,u_0) \ \ a.e \ \ in \ \ \Omega \ \ and \ \ strongly \ \ in \ \ L^1(\Omega).$$

(3.1.5)

Let us now consider the approximate problem:

$$\begin{cases} u_n \in K_\psi, \\ \int_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t}, (u_n - v)\right\rangle_{W^{-1,p'}(\Omega),W^{1\ p}(\Omega)} dxdt + \int_Q a(x,t,u_n,Du_n)D(u_n-v)dxdt \\ +\int_Q H_n(x,t,u_n,Du_n)(u_n-v)dxdt \leq \int_Q f_n(u_n-v)dxdt, \\ \forall v \in K_\psi. \end{cases}$$

(3.1.6)

Note that $H_n(x,t,s,\xi)$ satisfies the following conditions

$$|H_n(x,t,s,\xi)| \leq H(x,t,s,\xi) \ \ and \ \ |H_n(x,t,s,\xi)| \leq n.$$

For all $u,v \in L^p(0,T;W_0^{1\ p}(\Omega))$.
Moreover, since $f_n \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ then, for fixed $n \in \mathbb{N}$ the approximate problem (3.1.7) has at least one solution (see e.g [14]).

A priori estimate.

Let $v = u_n - \eta\exp(G(u_n))T_k(u_n^+ - \psi^+)$**,** where $G(s) = \int_0^s \frac{g(t)}{\alpha}\, dt$ **(**the function g appears in (3.1.8)**)** and $\eta \geq 0$. Since $v \in L^p(0,T;W_0^{1\ p}(\Omega))$ and for η small enough, we have $v \geq \psi$, thus v is admissible test function in (3.1.9), then

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\eta\exp(G(u_n))T_k(u_n^+ - \psi^+)dxdt$$

$$+\int_Q a(x,t,u_n,Du_n)D(\eta\exp(G(u_n))T_k(u_n^+-\psi^+))dxdt$$

$$+\int_Q H_n(x,t,u_n,Du_n)\eta\exp(G(u_n))T_k(u_n^+-\psi^+)dxdt$$

$$\leq \int_Q f_n\eta\exp(G(u_n))T_k(u_n^+ - \psi^+)dxdt$$

Then

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))T_k(u_n^+ - \psi^+)dxdt$$

$$+\int_Q a(x,t,u_n,Du_n)Du_n\frac{g(u_n)}{\alpha}\exp(G(u_n))T_k(u_n^+-\psi^+)dxdt$$

$$+\int_Q a(x,t,u_n,Du_n)D(T_k(u_n^+-\psi^+))\exp(G(u_n))dxdt$$

$$\leq \int_Q -H_n(x,t,u_n,Du_n)\exp(G(u_n))T_k(u_n^+-\psi^+)dxdt+\int_Q f_n\exp(G(u_n))T_k(u_n^+-\psi^+)dxdt.$$

$$\leq \int_Q \gamma(x,t)\exp(G(u_n))T_k(u_n^+-\psi^+)dxdt+\int_Q g(u_n)\,|Du_n|^p\exp(G(u_n))T_k(u_n^+-\psi^+)dxdt$$

$$+\int_Q f_n\exp(G(u_n))T_k(u_n^+ - \psi^+)dxdt,$$

In view of (2.1.4) we obtain

$$\int_\Omega B_{k,G}^n(x,u_n(\tau))dx+\int_Q a(x,t,u_n,Du_n)D(T_k(u_n^+-\psi^+))\exp(G(u_n))dxdt$$

$$\leq \int_Q \gamma(x,t)\exp(G(u_n))T_k(u_n^+-\psi^+)dxdt+\int_Q f_n\exp(G(u_n))T_k(u_n^+-\psi^+)dxdt$$

$$+\int_\Omega B_{k,G}^n(x,u_{0n})dx, \quad (3.2.1)$$

where
$B_{k,G}^n(x,r) = \int_0^r T_k(s^+ - \psi^+)\exp(G(s))\frac{\partial b_n(x,s)}{\partial s}ds.$
Due to the definition of $B_{k,G}^n$ we have

$$0 \leq \int_\Omega B_{k,G}^n(x,u_{0n})dx \leq k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\int_\Omega |b_n(x,u_{0n})|\,dx \leq kC\,\|b(x,u_0)\|_{L^1(\Omega)}.$$

(3.2.2)

Using (3.2.3) and $B_{k,G}^n(x,u_n) \geq 0$ and $G(u_n) \leq \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$ then we deduce that,

$$\int_Q a(x,t,u_n,Du_n)D(T_k(u_n^+-\psi^+))\exp(G(u_n))dxdt \leq c_1 k$$

where c_1 is a positive constant not depending on n.

Consequently, we have.

$$\int_{\{|u_n^+-\psi^+|\leq k\}} a(x,t,u_n,Du_n)Du_n^+\exp(G(u_n))\ dxdt$$

$$\leq \int_{\{|u_n^+-\psi^+|\leq k\}} a(x,t,u_n,Du_n)D\psi^+\exp(G(u_n))\ dxdt+c_1k.$$

Thanks to (2.1.4) for the left hand integral, (2.1.3) and Young's inequality for the right hand, we deduce

$$\int_{\{|u_n^+-\psi^+|\leq k\}} |Du_n^+|^p\ dxdt \leq c_2k + c_3.$$

(3.2.4)

Since
$\{(x,t) \in \Omega\times(0,T),\ |u_n^+| \leq k\} \subset \{(x,t) \in \Omega\times(0,T),\ |u_n^+ - \psi^+| \leq k + \|\psi^+\|_\infty\}$,
hence

$$\int_Q |DT_k(u_n^+)|^p\, dxdt = \int_{\{|u_n^+|\le k\}} |Du_n^+|^p\, dxdt \le \int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}} |Du_n^+|^p\, dxdt$$

moreover, (3.2.5) implies that,

$$\int_Q |DT_k(u_n^+)|^p\, dxdt \le c_4 k \quad \forall k > 0 \qquad (3.2.6)$$

where c_3 is a positive constant.
On the other hand, taking $v = u_n + \exp(-G(u_n))T_k(u_n^-)$ as test function in (3.1.6). Similarly we obtain

$$\int_Q |DT_k(u_n^-)|^p\, dxdt \le c_4 k \qquad (3.2.7)$$

where c_4 is a positive constant.
Combining (3.2.8) and (3.2.9), we conclude

$$\int_Q |DT_k(u_n)|^p\, dxdt \le ck \qquad (3.2.10)$$

where c is a constant positive.
We deduce from that above inequality (3.2.1) and (3.2.2) that

$$\int_\Omega B_k^n(x,u_n)dx \le k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)(\|f\|_{L^1(Q)} + \|b(x,u_0)\|_{L^1(\Omega)}) \equiv Ck. \qquad (3.2.11)$$

Then, $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1\ p}(\Omega))$, independently of n for any $k > 0$.
We deduce from that above inequality (3.2.1), (3.2.2) and (3.2.12) that

$$\int_\Omega B_{k,G}^n(x,u_n(\tau))dx \le C\ k. \qquad (3.2.13)$$

Now we turn to prove the almost every convergence of u_n and $b_n(x,u_n)$.
Consider now a function non decreasing $\omega_k \in C^2(\mathbb{R})$ such that $\omega_k(s) = s$ for $|s| \le \frac{k}{2}$ and $\omega_k(s) = k$ for $|s| \ge k$. Multiplying the approximate equation by $\omega_k'(u_n)$, we get

$$\frac{\partial B_k^n(x,u_n)}{\partial t} - div(a(x,t,u_n,Du_n)\omega_k'(u_n)) + a(x,t,u_n,Du_n)\omega_k''(u_n)Du_n$$

$$+H_n(x,t,u_n,Du_n)\omega_k'(u_n) = f_n\omega_k'(u_n) \qquad (3.2.14)$$

where $B_k^n(x,z) = \int_0^z \frac{\partial b_n(x,s)}{\partial s}\omega_k'(s)ds.$

As a consequence of (3.2.15), we deduce that $\omega_k(u_n)$ is bounded in $L^p(0,T;W_0^{1\ p}(\Omega))$ and $\frac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. Due to the properties of ω_k and (2.1.2), we conclude that $\frac{\partial \omega_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$, which implies that $\omega_k(u_n)$ is compact in $L^1(Q)$.
Due to the choice of ω_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converges almost everywhere to some measurable function v in Q. Thus by using the same argument as in[4,5,21], we can show the following lemma.

Lemma 0.1 *Let u_n be a solution of the approximate problem (??). Then*

$$u_n \to u \quad a.e \quad in \quad Q, \qquad (1)$$

and

$$b_n(x,u_n) \to b(x,u) \quad a.e \quad in \quad Q, \qquad (2)$$

we can deduce from (??) that,

$$T_k(u_n) \rightharpoonup T_k(u) \quad weakly \quad in \quad L^p(0,T;W_0^{1\ p}(\Omega)) \qquad (3)$$

Which implies, by using (??), for all $k > 0$ that there exists a function $\Lambda_k \in \left(L^{p'}(Q)\right)^N$, such that

$$a(x,t,T_k(u_n),DT_k(u_n)) \rightharpoonup \Lambda_k \quad weakly \quad in \quad \left(L^{p'}(Q)\right)^N. \qquad (4)$$

We now establish that $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$.
Using (1) and passing to the limit - inf in (3.2.16) as n tends to $+\infty$, we obtain that $\frac{1}{k}\int_\Omega B_{k,G}(x,u(\tau))dx \le C$, for almost any τ in $(0,T)$. Due to the definition of $B_{k,G}(x,s)$ and the fact that $\frac{1}{k}B_{k,G}(x,u)$ converges pointwise to $\int_0^u sgn(s)\frac{\partial b(x,s)}{\partial s}\exp(G(s))ds \ge |b(x,u)|$, as k tends to $+\infty$, shows that $b(x,u)$ belong to $L^\infty(0,T;L^1(\Omega))$.

Strong convergence of truncation.
This step is devoted to introduce for $k \ge 0$ fixed a time regularization of the function $T_k(u)$ in order to perform the

monotonicity method. This kind of method has been first introduced by R.Landes (see Lemma 6 and proposition 3,p.230, and proposition 4, p.231, in [12].
Let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.
Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that w^i_μ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad |w^i_\mu| \le k,$$
$$w^i_\mu \to T_k(u) \quad in \quad L^p(0,T; W_0^{1\ p}(\Omega)), \tag{3.3.1}$$

as $\mu \to \infty$.
We will introduce the following function of one real variable s, which is define as:

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \le m \\ 0 & \text{if } |s| \ge m+1 \\ m+1-s & \text{if } m \le s \le m+1 \\ m+1+s & \text{if } -(m+1) \le s \le -m \end{cases}$$

where $m > k$.
Let $v = u_n - \eta \exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)$, v is a test function in (3.1.6). Then, we have

$$\int_{\{T_k(u_n)-w^i_\mu \ge 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w^i_\mu) h_m(u_n) dxdt$$
$$+ \int_{\{T_k(u_n)-w^i_\mu \ge 0\}} a(x,t,u_n,Du_n) D(T_k(u_n) - w^i_\mu) h_m(u_n) dxdt$$
$$- \int_{\{m \le u_n \le m+1\}} \exp(G(u_n)) a(x,t,u_n,Du_n) Du_n (T_k(u_n) - w^i_\mu)^+ dxdt$$
$$\le \int_Q \gamma(x,t) \exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n) dxdt$$
$$+ \int_Q f_n \exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n) dxdt. \tag{3.3.2}$$

Observe that

$$\int_{\{m \le u_n \le m+1\}} \exp(G(u_n)) a(x,t,u_n,Du_n) Du_n (T_k(u_n) - w^i_\mu)^+ dxdt$$
$$\le 2k \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \int_{\{m \le u_n \le m+1\}} a(x,t,u_n,Du_n) Du_n dxdt.$$

We prove the following Lemma:

Lemma 0.1 *Let u_n be a solution of the approximate problem (**??**). Then*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m \le |u_n| \le m+1\}} a(x,t,u_n,Du_n) Du_n dxdt = 0 \tag{1}$$

Proof. Considering the following function $v = u_n - \eta \exp(G(u_n))T_1(u_n - T_m(u_n))^+$,
for m large enough and η small enough, we can deduce that $v \ge \psi$, and since $v \in L^p(0,T; W_0^{1\ p}(\Omega))$, v is a test function in (3.1.6). Then, we obtain,

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n)T_1(u_n - T_m(u_n))^+ dxdt$$
$$+ \int_Q a(x,t,u_n,Du_n) D(\exp(G(u_n)T_1(u_n - T_m(u_n))^+)\, dxdt$$
$$+ \int_Q H_n(x,t,u_n,Du_n) \exp(G(u_n)T_1(u_n - T_m(u_n))^+\, dxdt$$
$$\le \int_Q f_n \exp(G(u_n)T_1(u_n - T_m(u_n))^+\, dxdt.$$

Which gives, by setting
$B_n^m(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} \exp(G(s))T_1(s - T_m(s))^+ ds$,
and from the growth condition(2.2.1),

$$\int_\Omega B_n^m(x,u_n)(T)dx + \int_Q a(x,t,u_n,Du_n) \frac{g(u_n)}{\alpha} \exp(G(u_n)T_1(u_n - T_m(u_n))^+\, dxdt$$
$$+ \int_Q a(x,t,u_n,Du_n) \exp(G(u_n)DT_1(u_n - T_m(u_n))^+\, dxdt$$
$$\le \int_Q g(u_n)\, |Du_n|^p \exp(G(u_n)T_1(u_n - T_m(u_n))^+\, dxdt$$

$$+\int_Q (f_n+\gamma(x,t))\exp(G(u_n)T_1(u_n-T_m(u_n))^+\,dxdt+\int_\Omega B_n^m(x,u_0)dx,$$

which, thanks to (2.1.4) , gives:

$$\int_\Omega B_n^m(x,u_n)(T)dx+\int_Q a(x,t,u_n,Du_n)\exp(G(u_n)DT_1(u_n-T_m(u_n))^+\,dxdt$$

$$\leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\int_Q (f_n+\gamma(x,t))T_1(u_n-T_m(u_n))^+\,dxdt+\int_{|u_0|>m}|b(x,u_0)|\,dx\right],$$

(3.3.3)

Since $B_n^m(x,u_n)(T)\geq 0$, then by Lebesgue's theorem the right hand side goes to zero as n and m tend to infinity.
Therefore, passing to the limit first in n, then m, we obtain from (3.3.4)

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt=0.$$

(3.3.5)

On the other hand, consider the test function $v=u_n+\exp(-G(u_n))T_1(u_n-T_m(u_n))^-$ in (3.1.6) is cleary admissible, then

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\leq u_n\leq -m\}} a(x,t,u_n,Du_n)Du_n dxdt=0.$$

(3.3.6)

Thus Lemma 0.1 follows from (3.3.7) and (3.3.8).
Thanks to Lemma 0.1 the third integral tend to zero as n and m tend to infinity, and by Lebesgue's theorem, we deduce that the right hand side converge to zero as n, m and μ tend to infinity. Since

$$(T_k(u_n)-w_\mu^i)^+h_m(u_n)\rightharpoonup(T_k(u)-w_\mu^i)^+h_m(u)\ \ weakly*\ in\ L^\infty(Q),\ as\ n\to\infty\ and$$

$$(T_k(u)-w_\mu^i)^+h_m(u)\rightharpoonup 0\ \ weakly*\ \ in\ \ L^\infty(Q)\ \ as\ \ \mu\to\infty.$$

Let $\varepsilon_l(n,m,\mu,i)$ $\ l=1,...,n$ various functions tend to zero as n, m, i and μ tend to infinity.
The very definition of the sequence w_μ^i makes it possible to establish the following lemma.

Lemma 0.1 *For $k\geq 0$ we have*

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}\frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt\geq\varepsilon(n,m,\mu,i)$$

(1)

Proof: this lemma is proved in [19].
On the other hand, the second term of left hand side of (3.3.2) reads as

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt$$

$$=\int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|\leq k\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt$$

$$-\int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|\geq k\}} a(x,t,u_n,Du_n)Dw_\mu^i h_m(u_n)dxdt.$$

Since $m>k,\quad h_m(u_n)=0$ on $\{|u_n|\geq m+1\}$, one has

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt$$

$$=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt$$

$$-\int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|\geq k\}} a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))Dw_\mu^i h_m(u_n)dxdt=J_1+J_2.$$

(3.3.9)

In the following we pass to the limit in (3.3.10) : first we let n tend to $+\infty$, then μ and finally m, tend to $+\infty$. Since $a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))$ is bounded in $\left(L^{p'}(Q)\right)^N$, we have that

$$a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))h_m(u_n)\chi_{\{|u_n|>k\}}\to\Lambda_m\chi_{\{|u|>k\}}h_m(u)$$

strongly in $\left(L^{p'}(Q)\right)^N$ as n tends to infinity, it follows that

$$J_2=\int_{\{T_k(u)-w_\mu^i\geq 0\}}\Lambda_m Dw_\mu^i h_m(u)\chi_{\{|u|>k\}}dxdt+\varepsilon(n)$$

$$=\int_{\{T_k(u)-w_\mu^i\geq 0\}}\Lambda_m(DT_k(u)_\mu-e^{-\mu t}DT_k(\psi_i))h_m(u)\chi_{\{|u|>k\}}dxdt+\varepsilon(n).$$

By letting $\mu\to+\infty$, implies that

$$J_2=\int_Q\Lambda_m DT_k(u)dxdt+\varepsilon(n,\mu).$$

Using now the term J_1 of (3.3.11) one can easily show that

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt$$

$$=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times\left[DT_k(u_n)-DT_k(u)\right]h_m(u_n)dxdt$$

$$+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),DT_k(u))(DT_k(u_n)-DT_k(u))h_m(u_n)dxdt$$

$$+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),DT_k(u_n))DT_k(u)h_m(u_n)dxdt$$

$$-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),DT_k(u_n))Dw_\mu^i h_m(u_n)dxdt=K_1+K_2+K_3+K_4.$$

(3.3.12)

We shall go to the limit as n and $\mu\to+\infty$ in the three integrals of the last side.

Starting with K_2, we have by letting $n\to+\infty$

$$K_2=\varepsilon(n). \qquad (3.3.13)$$

About K_3, we have by letting $n\to+\infty$ and using (4)

$$K_3=\varepsilon(n). \qquad (3.3.14)$$

For what concerns K_4 we can write

$$K_4=-\int_{\{T_k(u)-w_\mu^i\geq 0\}}\Lambda_k Dw_\mu^i h_m(u)dxdt+\varepsilon(n),$$

By letting $\mu\to+\infty$, implies that

$$K_4=-\int_Q\Lambda_k DT_k(u)dxdt+\varepsilon(n,\mu).$$

(3.3.15)

We then conclude that

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),DT_k(u_n))\nabla(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt$$

$$=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times\left[DT_k(u_n)-DT_k(u)\right]h_m(u_n)dxdt+\varepsilon(n,\mu).$$

On the other hand, we have

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times\left[DT_k(u_n)-DT_k(u)\right]dxdt$$

$$=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times\left[DT_k(u_n)-DT_k(u)\right]h_m(u_n)dxdt$$

$$+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),DT_k(u_n))(DT_k(u_n)-DT_k(u))(1-h_m(u_n))dxdt$$

$$-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),DT_k(u))(DT_k(u_n)-DT_k(u))(1-h_m(u_n))dxdt.$$

(3.3.16)

Since $h_m(u_n)=1 \quad in \quad \{|u_n|\leq m\}$ and $\{|u_n|\leq k\}\subset\{|u_n|\leq m\}$ for m large enough, we deduce from (3.3.17) that

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times\left[DT_k(u_n)-DT_k(u)\right]dxdt$$

$$=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times\left[DT_k(u_n)-DT_k(u)\right]h_m(u_n)dxdt$$

$$+\int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|>k\}}a(x,t,T_k(u_n),DT_k(u))DT_k(u)(1-h_m(u_n))dxdt.$$

It is easy to see that the last terms of the last equality tend to zero as $n\to+\infty$, which implies that

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$
$$\times[DT_k(u_n)-DT_k(u)]\,dxdt$$

$$=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times[DT_k(u_n)-DT_k(u)]\,h_m(u_n)dxdt+\varepsilon(n)$$

Combining (Lemma 0.11), (3.3.8), (3.3.9), (3.3.10) , (3.3.11) and (3.3.12) we obtain

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times[DT_k(u_n)-DT_k(u)]\,dxdt\leq\varepsilon(n,\mu,m)$$
(3.3.18)

To pass to the limit in (3.3.19) as $n, \quad and \quad m$ tend to infinity, we obtain

$$\lim_{n\to\infty}\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times[DT_k(u_n)-DT_k(u)]\,dxdt=0.$$
(3.3.20)

On the other hand, take
$v=u_n+\exp(-G(u_n)))(T_k(u_n)-w_\mu^i)^-h_m(u_n)$.
This is a test function admissible in (3.1.6).
Similarly, we can deduce as in (3.3.21) that

$$\lim_{n\to\infty}\int_{\{T_k(u_n)-w_\mu^i\leq 0\}}[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]$$

$$\times[DT_k(u_n)-DT_k(u)]\,dxdt=0.$$
(3.3.22)

Combining (3.3.23) and (3.3.24), we conclude

$$\lim_{n\to\infty}\int_Q[a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\times[DT_k(u_n)-DT_k(u)]\,dxdt=0.$$

(3.3.25)

Which implies that ,

$$T_k(u_n)\to T_k(u)\ \ strongly\ \ in\ \ L^p(0,T;W_0^{1\ p}(\Omega))\ \forall k.$$
(3.3.26)

Now, observe that we have, for every $\sigma>0$

$$meas\{(x,t)\in\Omega\times[0,T]:|Du_n-Du|>\sigma\}\leq meas\{(x,t)\in\Omega\times[0,T]:|Du_n|>k\}$$
$$+meas\{(x,t)\in\Omega\times[0,T]:|u|>k\}$$
$$+meas\{(x,t)\in\Omega\times[0,T]:|DT_k(u_n)-DT_k(u)|>\sigma\}$$

then as a consequence of (3.3.27) we also have, that Du_n converges to Du in measure and therefore, always reasoning for subsequence,

$$Du_n\to Du\ \ a.e\ \ in\ \ Q. \qquad (3.3.28)$$

Which implies that,

$$a(x,t,T_k(u_n),DT_k(u_n))\rightharpoonup a(x,t,T_k(u),DT_k(u))\ \ in\ \ \left(L^{p'}(Q)\right)^N.$$
(3.3.29)

<u>Equi-integrability of the nonlinearity sequence.</u>

We shall now prove that
$H_n(x,t,u_n,Du_n)\to\ H(x,t,u,Du)$ strongly in $L^1(Q)$ by using Vitali's theorem.
Since $H_n(x,t,u_n,Du_n)\to\ H(x,t,u,Du)\ \ a.e\ \ in\ \ Q,$
Consider now a function $\rho_h(s)=\int_s^0 g(\nu)\chi_{\{\nu<-h\}}d\nu$.
On the one hand, let $v=u_n+\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$. Since $v\in L^p(0,T;W_0^{1\ p}(\Omega))$ and $v\geq\psi$, v is an admissible test function in (3.1.6). Then,

$$-\int_Q\frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\rho_h(u_n)dxdt+\int_Q a(u_n,Du_n)D\left(-\exp(-G(u_n))\rho_h(u_n)\right)dxdt+$$
$$+\int_Q H_n(u_n,Du_n)\left(-\exp(-G(u_n))\rho_h(u_n)\right)dxdt\leq\int_Q f_n\left(-\exp(-G(u_n))\rho_h(u_n)\right)dxdt.$$

Which implies that

$$\int_\Omega B_h^n(x,u_n)(T)dx+\int_Q a(u_n,Du_n)Du_n\frac{g(u_n)}{\alpha}\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}dsdxdt+$$
$$+\int_Q a(u_n,Du_n)Du_n\exp(-G(u_n))g(u_n)\chi_{\{u_n<-h\}}dxdt$$
$$\leq\int_Q\gamma(x,t)\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}dsdxdt$$
$$+\int_Q g(u_n)\,|Du_n|^p\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}dsdxdt$$
$$-\int_Q f_n\exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}dsdxdt+\int_\Omega B_h^n(x,u_{0n})dx,$$

where $B_h^n(x,z)=\int_0^z\frac{\partial b_n(x,s)}{\partial s}\exp(-G(s))(-\rho_h(s))ds$
using (2.1.4) and since
$\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds\leq\int_{-\infty}^{-h}g(s)ds$, we get

$$\int_Q a(u_n, Du_n)Du_n \exp(-G(u_n))g(u_n)\chi_{\{u_n<-h\}}dxdt$$
$$\leq \left(\int_{-\infty}^{-h} g(s)ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left(\|b(x,u_0)\|_{L^1(\Omega)}+\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}\right)$$
$$\leq \left(\int_{-\infty}^{-h} g(s)ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left(\|\gamma\|_{L^1(Q)}+\|f\|_{L^1(Q)}+\|b(x,u_0)\|_{L^1(\Omega)}\right)$$

using (2.1.4) , we obtain

$$\int_{\{u_n<-h\}} g(u_n)\,|Du_n|^p\,dxdt \leq C\int_{-\infty}^{-h} g(s)ds$$

and since $g \in L^1(\mathbb{R})$, see [7], we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} g(u_n)\,|Du_n|^p\,dxdt = 0.$$

(3.4.1)

On the other hand, let $M = \exp(-G(u_n))\int_0^{+\infty} g(s)ds$ and $h \geq M + \|\psi\|_{L^\infty(\Omega)}$. Consider $v = u_n - \exp(G(u_n))\int_0^{u_n} g(s)\chi_{\{s>h\}}ds$. Since $v \in L^p(0,T;W_0^{1\ p}(\Omega))$ and $v \geq \psi$, v is an admissible test function in (3.1.6). Then, similarly to (3.4.2), we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} g(u_n)\,|Du_n|^p\,dxdt = 0.$$

(3.4.3)

which implies, for h large enough and for a subset E of Q,

$$\lim_{meas(E)\to 0}\int_E g(u_n)\,|Du_n|^p\,dxdt \leq \|g\|_\infty \lim_{meas(E)\to 0}\int_E |DT_h(u_n)|^p\,dxdt + \int_{\{|u_n|>h\}} g(u_n)\,|Du_n|^p\,dxdt$$

then we conclude that $g(u_n)\,|Du_n|^p$ is equi - integrale. Thus we have obtained that $g(u_n)\,|Du_n|^p$ converge to $g(u)\,|Du|^p$ strongly in $L^1(Q)$. Consequently, by using (2.2.1) , we conclude that

$$H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)\ \ strongly\ \ in\ \ L^1(Q).$$

(3.4.4)

Passing to the limit.

Observe that for any fixed $m \geq 0$ **one has**

$$\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n = \int_Q a(x,t,u_n,Du_n)(DT_{m+1}(u_n)-DT_m(u_n))$$
$$= \int_Q a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))DT_{m+1}(u_n) - \int_Q a(x,t,T_m(u_n),DT_m(u_n))DT_m(u_n).$$

According to **(4**) and (3.3.17) , one is at liberty to pass to the limit as $n \to +\infty$ for fixed $m \geq 0$ and to obtain

$$\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt$$
$$= \int_Q a(x,t,T_{m+1}(u),DT_{m+1}(u))DT_{m+1}(u)dxdt - \int_Q a(x,t,T_m(u),DT_m(u))DT_m(u)dxdt.$$
$$= \int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Dudxdt.$$

(3.4.5)

Taking the limit as $m \to +\infty$ in (3.4.6) and using the estimate Lemma 0.1 show that u satisfies

$$\lim_{m\to+\infty}\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Dudxdt = 0$$

(3.4.7)

On the other hand, let $\varphi \in K_\psi \cap L^\infty(Q)$, $\frac{\partial\varphi}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ $with\ \varphi(x,T) = 0$ $such\ \ that\ \ S'(u)\varphi \in L^p(0,T;W_0^{1\ p}(\Omega))$ and Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let M be a positive real number such that supp$(S') \subset [-M,M]$. take $v = u_n - S'(u_n)\varphi$ as a test function in (3.1.6). We get,

$$\begin{cases} u_n \in K_\psi. \\ -\int_\Omega B_{S,n}(x,u_{n0})\varphi(x,0)dx - \int_0^T\int_\Omega \frac{\partial\varphi}{\partial t}B_{S,n}(x,u_n)dxdt \\ +\int_Q S'(u_n)a(x,t,u_n,Du_n)D\varphi dxdt + \int_Q S''(u_n)a(x,t,u_n,Du_n)Du_n\varphi dxdt \\ +\int_Q H(x,t,u_n,Du_n)S'(u_n)\varphi dxdt \leq \int_Q fS'(u_n)\varphi dxdt. \end{cases}$$

(3.4.8)

Where $B_{S,n}(x,r) = \int_0^r \frac{\partial b_n(x,l)}{\partial s}S'(l)dl.$

In what follows we pass to the limit as $n \to +\infty$ in each term of (3.4.9).

- Since S is bounded and continuous, $u_n \to u\ \ a.e\ \ in\ \ Q$ implies that $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e in Q and $L^\infty\ \ weak-*$. Then $\frac{\partial B_S^n(x,u_n)}{\partial t}$ converges to $\frac{\partial B_S(x,u)}{\partial t}$ in $D'(Q)$ as n

tends to $+\infty$.
$\frac{\partial\varphi}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega))$. Then

$$\int_Q \frac{\partial\varphi}{\partial t} B_{S,n}(x,u_n)dxdt \to \int_Q \frac{\partial\varphi}{\partial t} B_S(x,u)dxdt. \quad (3.4.10)$$

• Limit of $S'(u_n)a_n(x,t,u_n,Du_n)$.
Since supp$(S') \subset [-M,M]$, we have for $n \geq M$

$$S'(u_n)a_n(x,t,u_n,Du_n) = S'(u_n)a(x,t,T_M(u_n),DT_M(u_n)) \ \ a.e \ \ in \ \ Q.$$

The pointwise convergence of $u_n \ \ to \ \ u$ and (4) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(x,t,u_n,Du_n) \rightharpoonup S'(u)a(x,t,T_M(u),DT_M(u)) \ \ in \ \ \left(L^{p'}(Q)\right)^N, \quad (3.4.11)$$

as n tends to $+\infty$. $S'(u)a(x,t,T_M(u),DT_M(u))$ has been denoted by $S'(u)a(x,t,u,Du)$.

• Limit of $S''(u_n)a(x,t,u_n,Du_n)Du_n$.
As far as the 'energy' term

$$S''(u_n)a(x,t,u_n,Du_n)Du_n = S''(u_n)a(x,t,T_M(u_n),DT_M(u_n))DT_M(u_n) \ \ a.e \ \ in \ \ Q.$$

The pointwise convergence of $S'(u_n) \ \ to \ \ S'(u)$ and (4) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a_n(x,t,u_n,Du_n)Du_n \rightharpoonup S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) \ \ weakly \ in \ L^1(Q). \quad (3.4.12)$$

Recall that
$S''(u)\varphi a(x,t,T_M(u),DT_M(u))DT_M(u) = S''(u)\varphi a(x,t,u,Du)Du \ \ a.e \ \ in \ \ Q.$

• Limit of $S'(u_n)\varphi H_n(x,t,u_n,Du_n)$.
Since supp$(S') \subset [-M,M]$ and (3.4.13), we have

$$\int_Q S'(u_n)\varphi H_n(x,t,u_n,Du_n)dxdt \to \int_Q S'(u)\varphi H(x,t,u,Du)dxdt, \quad (3.4.14)$$

as n tends to $+\infty$.

• Limit of $S'(u_n)\varphi f_n$.
Since $u_n \to u$ a.e in Q, we have

$\int_Q S'(u_n)\varphi f_n dxdt \to \int_Q S'(u) f dxdt$ as $n \to +\infty$.

To this end, firstly remark that, S being bounded, $B^n_S(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly, the above considerations on the behavior of the terms of this equation show that $\frac{\partial B^n_S(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin's type lemma (see, e.g. [23]) implies that $B^n_S(x,u_n)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. It follows that on the one hand, $B^n_S(x,u_n)(t=0) = B^n_S(x,u^n_0)$ converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that $B_S(x,u)(t=0) = B_S(x,u_0)$ in Ω. we can pass to the limit in (2). This completes the proof of Theorem 0.1.

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