The Comparative Relation and Its Application in solving Fuzzy Linear Programming Problem

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Abstract

This paper considers linear programming problem whose objective function is fuzzy numbers vector. In order to solve this problem, we first present a new definition of comparative relation on the set of fuzzy numbers. Based on this definition, we state a method to compare fuzzy numbers directly and then, by the related theorems and lemmas, we build an algorithm to solve fuzzy presented problem.

Keywords: *Fuzzy linear programming, fuzzy objective function, triangular fuzzy number, comparative relation..*

1. Introduction

The problem of linear programming (LP) can be written as:

$$\begin{cases} Min/Max & z = CX \\ s.t. & AX \le B \\ & X \in R_+^n \end{cases}$$

in which:

- The vector of coefficients of the objective function: $C = (c_1, c_2, \ldots, c_n) \in R^n$;

- The vector of variables: $X = (x_1, x_2, \ldots, x_n)^T \in R_+^n$ i.e $x_j \in R$, $x_j \ge 0$, $j = 1,2,\ldots,n$;

- The matrix A is the $m \times n$ matrix of coefficients of the left-hand sides of the equalities:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

- The vector of right-hand sides of the equalities: $B = (b_1, b_2, \ldots, b_m)^T \in R^m$.

According to conventional approach, all the coefficients of LP must be defined clearly. However, the data in practical problem may only be uncertainly estimated so they are possible to be characterized with fuzzy numbers. That LP, in which at least one coefficient is a fuzzy number, is called fuzzy linear programming problem.

In this paper we consider a linear programming problem in which the coefficients defining the objective functions are given as fuzzy numbers. The aim of this paper is to propose a new definition of comparative relation on the set of fuzzy numbers. By a comparative relation and the related theorems and lemmas, fuzzy problem is transformed into crip linear programming problems and then it will be settled by available tools such as Lingo or Solver.

This paper is organized in 5 sections. In section 2, we first repeat the basic definitions of fuzzy number, triangle fuzzy number and fuzzy arithmetic operations. After that, we build the definitions of comparative relation on the set of fuzzy numbers. Based on this definition, we introduce a method to compare the two fuzzy numbers.

In the section 3 of this paper, we present the model of linear programming problem with the coefficients of objective function that is represented by the triangular fuzzy numbers and show the algorithm to transform solving this problems into crip linear programming problems based on the related theorems and lemmas.

In section 4, algorithm is illustrated by solving two numerical examples and conclusions are drawn in section 5.

2. Preliminaries

Definition 2.1 ([3], [4], [5]):
We represent an arbitrary fuzzy number by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right)$, $0 \le r \le 1$, which satisfy the following requirements:

- $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1];

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- $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0,1];
- $\underline{u}(r), \overline{u}(r)$ are right continuous in 0;
- $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

A crip number ω is simply represented by $\underline{u}(r) = \overline{u}(r) = \omega$, $0 \le r \le 1$.
The set of all arbitrary fuzzy numbers is denoted by *FN*.

Definition 2.2 ([3], [4], [5]): For arbitrary fuzzy numbers $\tilde{x} = (\underline{x}(r), \overline{x}(r))$, $\tilde{y} = (\underline{y}(r), \overline{y}(r))$ and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as:

- $\tilde{x} + \tilde{y} = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$.
- $k\tilde{x} = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \ge 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$
- $\tilde{x} - \tilde{y} = (\underline{x}(r) - \overline{y}(r), \overline{x}(r) - \underline{y}(r))$.

Definition 2.3 ([3], [4], [5], [6]): Two fuzzy numbers $\tilde{x} = (\underline{x}(r), \overline{x}(r))$, $\tilde{y} = (\underline{y}(r), \overline{y}(r))$ are equal, i.e, $\tilde{x} \cong \tilde{y}$ if and only if: $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$, $0 \le r \le 1$.

Lemma 2.1: Let $x, y \in R$, if $x = y$ then $x \cong y$.

Proof: Proof is clear.

Afterwards, if $\tilde{x} \cong \tilde{y}$ then we write $\tilde{x} = \tilde{y}$, in brief.

Definition 2.4 ([4], [6]):
The fuzzy number $\tilde{a} = (a - \alpha + \alpha r, a + \beta - \beta r)$; $0 \le r \le 1$, $a, \alpha, \beta \in R$ and $\alpha, \beta \ge 0$ is called Triangular Fuzzy Number with core *a*, lower limit $a - \alpha$ and upper limit $a + \beta$.

Symbolically, we write $\tilde{a} = (a; \alpha, \beta)$ where α and β are called left and right spread of $\tilde{a}$, respectively.
A crip number b is simply represented by $(b;0,0)$.
Let TFN be the set of all Triangular Fuzzy Number.

Let $\tilde{a} = (a; \alpha, \beta)$, $\tilde{b} = (b; \gamma, \lambda) \in TFN$ and $k \in R$. Then, the result of applying the definitions 2.2 and 2.3 on TFN as shown in the following:

- **Addition**: $\tilde{a} + \tilde{b} = (a + b; \alpha + \gamma, \beta + \lambda)$.
- **Scalar Multiplication**:

$$k\tilde{a} = \begin{cases} (ka; k\alpha, k\beta), & k \ge 0, \\ (ka, -k\beta, -k\alpha), & k < 0. \end{cases}$$

- **Image of** $\tilde{a}$: $-\tilde{a} = (-a; \beta, \alpha)$.
- **Subtraction**: $\tilde{a} - \tilde{b} = \tilde{a} + (-\tilde{b})$.
- **Equality**: $\tilde{a} = \tilde{b}$ if and only if $a = b$ and $\alpha = \gamma$ and $\beta = \lambda$.

A key question that may be encountered in solving LP with fuzzy number in the objective function is that how to find the optimal value. The answer is related to the problem of ranking fuzzy numbers.

A simple approach to ordering of the elements of *FN* is to use a ranking function $g(.): FN \to R$ which maps each fuzzy number into the real line, where a natural order exists, and defining order on *FN* by:

- $g(\tilde{a}) < g(\tilde{b}) \Rightarrow \tilde{a}$ is less than $\tilde{b}$;
- $g(\tilde{a}) > g(\tilde{b}) \Rightarrow \tilde{a}$ is greater than $\tilde{b}$;
- $g(\tilde{a}) = g(\tilde{b}) \Rightarrow \tilde{a}$ is equal to $\tilde{b}$.

Many ranking functions have been proposed so far. For any arbitrary fuzzy number $\tilde{a} = (\underline{a}(r), \overline{a}(r))$, L. Alizadeh, T.Allahviranloo, F. Hosseinzadeh Lotfi, M. Kh. Kiasary and N. A. Kiani [4] use ranking function:

$$g(\tilde{a}) = 1/2 \int_{[0,1]} (\underline{a}(r)) + \int_{[0,1]} (\overline{a}(r)).$$

For triangular fuzzy number $\tilde{a} = (a; \alpha, \beta)$, this is reduced to $g(\tilde{a}) = a + 1/4(\alpha + \beta)$.

In [6], Behrouz Kheirfam presented:

$$g(\tilde{a}) = \frac{(a - \alpha) + 2a + (a + \beta)}{4} = a + \frac{\beta - \alpha}{4}.$$

In spite of being expressed in many different formulas, these ranking functions are in contradiction with the definition about equality of fuzzy numbers (def. 2.3). For example, with two triangular fuzzy numbers $\tilde{x} = (15;1,5)$ and $\tilde{y} = (14;1,9)$ we have $\tilde{x} = \tilde{y}$ by using ranking functions not only presented in [4] but also in [6]. This result contradicts the definition 2.3, in which if $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$.

In this article, we want to find a way to compare fuzzy numbers directly too. It should be noted that, if discovered then *FN* becomes a total ordered set. In the case of crip sets, that means for every pair *x* and *y* has a total ordered

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relation. In the case of fuzzy number set, we think, it has a comparative relation. So we build the following definition:

Definition 2.5: Comparative Relation
A binary relation Ψ on FN is a comparative relation if it satisfies the following requirements:

a. $\forall \tilde{a}, \tilde{b} \in FN$ we have the usual trichotomy condition that exactly one of: $\tilde{a} \underset{\Psi}{<} \tilde{b}$ OR $\tilde{a} \underset{\Psi}{>} \tilde{b}$ OR $\tilde{a} = \tilde{b}$ holds; in which $\tilde{a} \underset{\Psi}{<} \tilde{b}$ means that ($\tilde{a}$ is smaller than $\tilde{b}$) OR ($\tilde{b}$ is greater than $\tilde{a}$) in structure (FN, Ψ). (totality)

b. Let $\tilde{a} \underset{\Psi}{\leq} \tilde{b}$ if and only if $\tilde{a} \underset{\Psi}{<} \tilde{b}$ OR $\tilde{a} = \tilde{b}$. We have $\forall \tilde{a} \in FN$, $\tilde{a} \underset{\Psi}{\leq} \tilde{a}$; (reflexivity)

c. $\forall \tilde{a}, \tilde{b} \in FN$, $\tilde{a} \underset{\Psi}{\leq} \tilde{b}$ and $\tilde{b} \underset{\Psi}{\leq} \tilde{a}$ imply that $\tilde{a} = \tilde{b}$ i.e. $\underline{a}(r) = \underline{b}(r)$ and $\overline{a}(r) = \overline{b}(r)$, $0 \leq r \leq 1$; (antisymmetry)

d. $\forall \tilde{a}, \tilde{b}$ and $\tilde{c} \in FN$, $\tilde{a} \underset{\Psi}{\leq} \tilde{b}$ and $\tilde{b} \underset{\Psi}{\leq} \tilde{c}$ imply that $\tilde{a} \underset{\Psi}{\leq} \tilde{c}$; (transitivity)

e. $\forall a, b \in R$, if $a \Theta b$ then $a \underset{\Psi}{\Theta} b$ where $\Theta = \{<, \leq, =, >, \geq\}$ (compatibility with order relation on the set of real number).

Suppose $\tilde{a} = (a; \alpha, \beta)$, $\tilde{b} = (b; \gamma, \lambda)$ are triangular fuzzy numbers ($\tilde{a}, \tilde{b} \in TFN$). Let $\Re$ is binary relation on TFN ($\Re \subseteq TFN^2$) where $\tilde{a} \underset{\Re}{<} \tilde{b}$ will be defined by :

1. $a < b$
or
2. $a = b$ and $\beta < \lambda$
or
3. $a = b$ and $\beta = \lambda$ and $\alpha < \gamma$

Theorem 2.1: $\Re$ is a comparative relation on TFN.
Proof: The theorem will be demonstrated by the following lemmas:

Lemma 2.2: For any $\tilde{a} = (a; \alpha, \beta), \tilde{b} = (b; \gamma, \lambda) \in TFN$ exactly one of the three statements $\tilde{a} \underset{\Re}{<} \tilde{b}$, $\tilde{a} = \tilde{b}$, $\tilde{a} \underset{\Re}{>} \tilde{b}$ holds.
Proof: See figure 1:

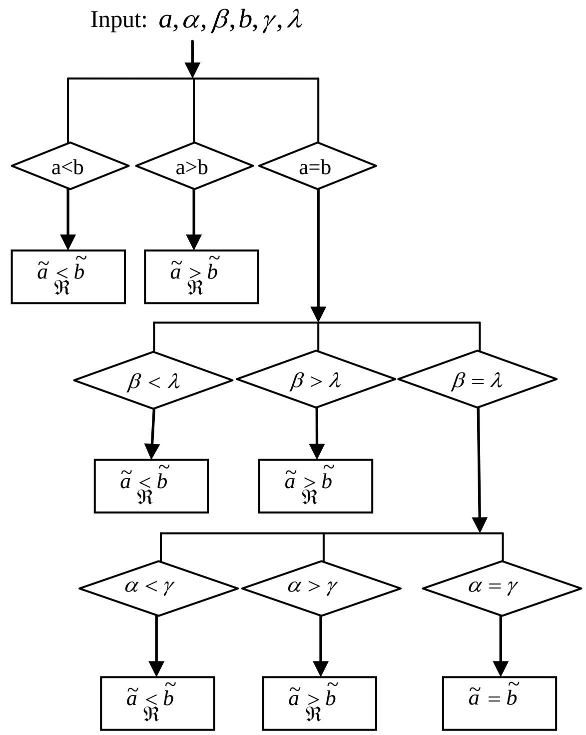

Fig. 1 The totality of $\Re$.

Lemma 2.3:
- If **(** $\tilde{a} \underset{\Re}{\leq} \tilde{b}$ **)** then $a \leq b$**.**
- If ($\tilde{a} \underset{\Re}{\leq} \tilde{b}$ and $a = b$**)** then $\beta \leq \lambda$.
- If ($\tilde{a} \underset{\Re}{\leq} \tilde{b}$ and $a = b$ and $\beta = \lambda$) then $\alpha \leq \gamma$.

Proof: Proof is clear.

Lemma 2.4: The relation $\underset{\Re}{\leq}$ satisfies three properties reflexivity, antisymmetry and transitivity.
Proof:
- It is clear that $\tilde{a} \leq \tilde{a}$ (reflexivity).
- Consider any pair $\tilde{a}, \tilde{b} \in TFN$ where $\tilde{a} \underset{\Re}{\leq} \tilde{b}$ and $\tilde{b} \underset{\Re}{\leq} \tilde{a}$:

+ Based on lemma 2.3, we have: $\tilde{a} \underset{\Re}{\leq} \tilde{b} \Rightarrow a \leq b$ and $\tilde{b} \underset{\Re}{\leq} \tilde{a} \Rightarrow b \leq a$. This establishes that $a = b$.

+ From $\tilde{a} \underset{\Re}{\leq} \tilde{b}$ and $a = b \Rightarrow \beta \leq \lambda$. Similarly, with $\tilde{b} \underset{\Re}{\leq} \tilde{a}$ and $b = a \Rightarrow \lambda \leq \beta$. This implies that $\lambda = \beta$.

+ From $\tilde{a} \underset{\Re}{\leq} \tilde{b}$ and $a = b$ and $\beta = \lambda \Rightarrow \alpha \leq \gamma$. On the other hand, from $\tilde{b} \underset{\Re}{\leq} \tilde{a}$ and $b = a$ and $\lambda = \beta \Rightarrow \gamma \leq \alpha$. Then we have $\gamma = \alpha$.

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Thus, from assumption $\tilde{a} \underset{\Re}{\leq} \tilde{b}$ and $\tilde{b} \underset{\Re}{\leq} \tilde{a}$ we have shown that $a = b$ and $\alpha = \gamma$ and $\beta = \lambda$. Consequently, $\tilde{a} = \tilde{b}$ (antisymmetry).

- Suppose that $\tilde{a} \underset{\Re}{\leq} \tilde{b}$ *and* $\tilde{b} \underset{\Re}{\leq} \tilde{c}$ where $\tilde{c} = (c; \omega, \xi) \in TFN$. By using the lemma 2.3, we have the inequality $a \leq b \leq c$. This implies that $a < c$ or $a = b = c$ holds.

+ Case 1: If $a < c$ then $\tilde{a} \underset{\Re}{<} \tilde{c}$.

+ Case 2: If $a = b = c$. Based on the lemma 2.3, we have $\beta \leq \lambda \leq \xi$. We also consider separately the following two cases: (2.1) $\beta < \xi$ and (2.2) $\beta = \lambda = \xi$.

* Case 2.1: If $\beta < \xi$. From $a = c$ and $\beta < \xi$ we have $\tilde{a} \underset{\Re}{<} \tilde{c}$.

* Case 2.2: If $\beta = \lambda = \xi$. According to the lemma 2.3, we have $\beta \leq \lambda \leq \xi$, so that $\beta < \xi$ or $\beta = \lambda = \xi$. Hence $\tilde{a} \underset{\Re}{\leq} \tilde{c}$.

And so we would say, in summary, if $\tilde{a} \underset{\Re}{\leq} \tilde{b}$ *and* $\tilde{b} \underset{\Re}{\leq} \tilde{c}$ then $\tilde{a} \underset{\Re}{\leq} \tilde{c}$ (transitivity). Proof is completed.

Lemma 2.5: $\Re$ is compatible with the order relation on set of real numbers.

Proof: Proof is clear.

From lemmas 2.2, 2.4 and 2.5 we have $\Re$ that is a comparative relation on *TFN* (Q.E.D).

3. Fuzzy linear programming problem

In this section, we present an application of comparative relation to solve linear programming problem where C is the vector of triangular fuzzy numbers. Based on the properties of comparative relation $\Re$, fuzzy problem is transformed into crip problems. After that, they will be settled by available tools such as Lingo or Solver.

Definition 3.1: A linear programming problem where the objective function is represented by triangular fuzzy numbers and maximum in accordance with $\Re$ is defined as follows:

$$\begin{cases} \underset{\Re}{Max} & \tilde{z} = \tilde{C}X & \\ s.t. & AX\{\leq, =, \geq\}B & (1) \\ & X \in R_+^n & (2) \end{cases} \qquad \text{(FLP)}$$

which $\tilde{C} = (\tilde{c}_1, .., \tilde{c}_n) \in TFN^n$; $\tilde{c}_j = (c_j; \alpha_j, \beta_j)$, $j = 1,2,\ldots,n$

Lemma 3.1: Let $C = (c_1, \ldots, c_n) \in R^n$, $\Phi = (\alpha_1, \ldots, \alpha_n) \in R^n$ and $\Omega = (\beta_1, \ldots, \beta_n) \in R^n$ is core, left spread and right spread vector of $\tilde{C}$ respectiverly. We have:

$$\tilde{z} = \tilde{C}X = (CX; \Phi X, \Omega X).$$

Proof:

$$\tilde{z} = \tilde{C}X = \sum_{j=1}^{n} \tilde{c}_j x_j = \tilde{c}_1 x_1 + \tilde{c}_2 x_2 + .. + \tilde{c}_n x_n$$
$$= (c_1; \alpha_1, \beta_1)x_1 + .. + (c_n; \alpha_n, \beta_n)x_n$$
$$= (c_1x_1; \alpha_1x_1, \beta_1x_1) + .. + (c_nx_n; \alpha_nx_n, \beta_nx_n)$$
$$= (c_1x_1 + .. + c_nx_n; \alpha_1x_1 + \ldots + \alpha_nx_n, \beta_1x_1 + .. + \beta_nx_n)$$
$$= (\sum_{j=1}^{n} c_j x_j; \sum_{j=1}^{n} \alpha_j x_j, \sum_{j=1}^{n} \beta_j x_j) = (CX; \Phi X, \Omega X) \text{ (Q.E.D).}$$

Definition 3.2: We say that vector $X = (x_1, x_2, \ldots, x_n)^T$ is a feasible solution to (FLP) if X satisfies the constraints (1) and (2) of the problem (FLP).

Definition 3.3: A feasible solution $X^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is an optimal solution for (FLP), if for all feasible solutions X, we have $\tilde{z}_X = \tilde{C}X \underset{\Re}{\leq} \tilde{z}_{X^*} = \tilde{C}X^*$.

$\tilde{z}_{X^*}$ is called the optimal value for (FLP) and also written as $\tilde{z}_{FLP}^*$.

Definition 3.4: The following are core, right spread and left spread problems of (FLP), called (I), (II) and (III) for short:

$$\begin{cases} Max & z_1 = CX & \\ s.t. & AX\{\leq, =, \geq\}B & (1) \\ & X \in R_+^n & (2) \end{cases} \qquad \text{(I)}$$

and

$$\begin{cases} Max & z_2 = \Omega X & \\ s.t. & AX\{\leq, =, \geq\}B & (1) \\ & X \in R_+^n & (2) \\ & CX = z_1^* & (3) \end{cases} \qquad \text{(II)}$$

where z_1^* is the optimal value of (I) and

$$\begin{cases} Max & z_3 = \Phi X & \\ s.t. & AX\{\leq, =, \geq\}B & (1) \\ & X \in R_+^n & (2) \\ & CX = z_1^* & (3) \\ & \Omega X = z_2^* & (4) \end{cases} \qquad \text{(III)}$$

where z_2^* is the optimal value of (II).

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Let S_{FLP}, S_1, S_2, S_3 be the set of feasible solutions; E_{FLP}, E_1, E_2, E_3 be the set of optimal solutions for (FLP), (I), (II) and (III) respectiverly.

Remark 3.1: The inclusive relation between the set of feasible solutions and the set of optimal solutions:

- $E_{FLP} \subseteq S_{FLP}$.
- $E_1 \subseteq S_1$.
- $E_2 \subseteq S_2$.
- $E_3 \subseteq S_3$.

Lemma 3.2::

- $S_{FLP} = S_1$.
- $S_2 = E_1$.
- $S_3 = E_2$.

Proof: Proof is clear.

Theorem 3.1:

a. $E_{FLP} \subseteq E_1$.

b. $E_{FLP} \subseteq E_2$.

Proof:

a. In order to prove $E_{FLP} \subseteq E_1$ we consider the following two cases separately:

- Case 1: $E_{FLP} = \varnothing$. It is clear that $E_{FLP} \subseteq E_1$.

- Case 2: $E_{FLP} \neq \varnothing$. Since $E_{FLP} \subseteq S_{FLP} \Rightarrow S_{FLP} \neq \varnothing$

+ Let $X^*_{FLP} \in E_{FLP}$ and $X_{FLP} \in S_{FLP}$. By using definition 3.3 and lemma 3.1 we have $\tilde{z}_{X_{FLP}} \underset{\Re}{\leq} \tilde{z}_{X^*_{FLP}}$ That means $(CX_{FLP}; \Phi X_{FLP}, \Omega X_{FLP}) \underset{\Re}{\leq} (CX^*_{FLP}; \Phi X^*_{FLP}, \Omega X^*_{FLP})$.

Based on lemma 2.3, we have $CX_{FLP} \leq CX^*_{FLP}$ (*i*)

+ On the other hand, by remark 3.1 and lemma 3.2 we have $X^*_{FLP} \in E_{FLP} \subseteq S_{FLP} = S_1$; $X_{FLP} \in S_{FLP} = S_1$. That means both X^*_{FLP} và X_{FLP} are feasible solutions for (I) (*ii*)

+ From (*i*) and (*ii*) we have X^*_{FLP} is an optimal solution for (I), thus $X^*_{FLP} \in E_1$ which implies that $E_{FLP} \subseteq E_1$. (Q.E.D)

b. Similarly, in order to prove $E_{FLP} \subseteq E_2$, we have to show that $\forall X^*_{FLP} \in E_{FLP}$ then $X^*_{FLP} \in E_2$. Consider the two following cases:

- If $E_{FLP} = \varnothing \Rightarrow E_{FLP} \subseteq E_2$.
- If $E_{FLP} \neq \varnothing \Rightarrow \exists X^*_{FLP} \in E_{FLP}$.

+ As shown above, we have $X^*_{FLP} \in E_{FLP} \subseteq E_1 = S_2$, so $X^*_{FLP} \in S_2$ and $S_2 \neq \varnothing$. (*iii*).

+ Consider a feasible solution to (II): $X_2 \in S_2$, we have $CX_2 = z^*_1$.

+ By remark 3.1 and lemma 3.2 we have: $X_2 \in S_2 = E_1 \subseteq S_1 = S_{FLP}$ so $X_2 \in S_{FLP}$. Besides, X^*_{FLP} is an optimal solution for (*FLP*) therefore $\tilde{z}_{X_2} \underset{\Re}{\leq} \tilde{z}_{X^*_{FLP}}$ which means that:

$(CX_2; \Phi X_2, \Omega X_2) \underset{\Re}{\leq} (CX^*_{FLP}; \Phi X^*_{FLP}, \Omega X^*_{FLP})$

or

$(z^*_1; \Phi X_2, \Omega X_2) \underset{\Re}{\leq} (z^*_1; \Phi X^*_{FLP}, \Omega X^*_{FLP})$.

Thus $\Omega X_2 \leq \Omega X^*_{FLP}$ (*iv*) by using lemma 2.2.

+ By (*iii*) and (*iv*) we have X^*_{FLP} is an optimal solution for (II), that is, $X^*_{FLP} \in E_2$ (Q.E.D).

Corollary 1:

- If $E_1 = \varnothing$ then $E_{FLP} = \varnothing$.
- If $E_2 = \varnothing$ then $E_{FLP} = \varnothing$.

Theorem 3.2: $E_{FLP} = E_3$.

Proof: This theorem will be proved by showing that $E_{FLP} \subseteq E_3$ and $E_3 \subseteq E_{FLP}$

a. Proof $E_{FLP} \subseteq E_3$

- If $E_{FLP} = \varnothing$. Obviously $E_{FLP} \subseteq E_3$
- If $E_{FLP} \neq \varnothing$. Let $X^*_{FLP} \in E_{FLP}$.

+ By theorem 3.1 and lemma 3.2, we have $X^*_{FLP} \in E_{FLP} \subseteq E_2 = S_3$, so $X^*_{FLP} \in S_3$. (*v*)

+ Now if $X_3 \in S_3$ then X_3 have to satisfy constraints (3) and (4) so that $CX_3 = z^*_1$ and $\Omega X_3 = z^*_2$. Similarly, because $X^*_{FLP} \in S_3$ so $CX^*_{FLP} = z^*_1$ and $\Omega X^*_{FLP} = z^*_2$.

+ On the other hand, by lemma 3.2 and remark 3.1, we have: $X_3 \in S_3 = E_2 \subseteq S_2 = E_1 \subseteq S_1 = S_{FLP}$. That means X_3 is a feasible solution of (FLP). Hence $\tilde{z}_{X_3} \underset{\Re}{\leq} \tilde{z}_{X^*_{FLP}}$ or $(CX_3; \Phi X_3, \Omega X_3) \underset{\Re}{\leq} (CX^*_{FLP}; \Phi X^*_{FLP}, \Omega X^*_{FLP})$ which means $(z^*_1; \Phi X_3, z^*_2) \underset{\Re}{\leq} (z^*_1; \Phi X^*_{FLP}, z^*_2)$. So $\Phi X_3 \leq \Phi X^*_{FLP}$ (*vi*) by lemma 2.2

+ From (*v*) and (*vi*), we have X^*_{FLP} is an optimal solution of (III), so that $X^*_{FLP} \in E_3$ hence $E_{FLP} \subseteq E_3$ and proof is completed.

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IJCSI International Journal of Computer Science Issues, Vol. 9, Issue 4, No 2, July 2012
ISSN (Online): 1694-0814
www.IJCSI.org

b. Proof $E_3 \subseteq E_{FLP}$

- If $E_3 = \varnothing \Rightarrow E_3 \subseteq E_{FLP}$.
- If $E_3 \neq \varnothing$. Let $X_3^* \in E_3$. We know that $X_3^* \in E_3 \subseteq S_3$ so $CX_3^* = z_1^*$ and $\Omega X_3^* = z_2^*$.

+ By lemma 3.2 and remark 3.1 we have:

$X_3^* \in E_3 \subseteq S_3 = E_2 \subseteq S_2 = E_1 \subseteq S_1 = S_{FLP}$.

So X_3^* is a feasible solution of (FLP) . (*vii*)

+ Now if $X_{FLP} \in S_{FLP}$. We have:

$\tilde{z}_{X_{FLP}} = (CX_{FLP}; \Phi X_{FLP}, \Omega X_{FLP})$;

$\tilde{z}_{X_3^*} = (CX_3^*; \Phi X_3^*, \Omega X_3^*) = (z_1^*; \Phi X_3^*, z_2^*)$.

Consider the following two cases separately:

* Case 1: $X_{FLP} \notin E_{FLP} \Rightarrow CX_{FLP} < z_1^*$. By the definition about $\Re$, we have $\tilde{z}_{X_{FLP}} < \tilde{z}_{X_3^*}$. From (*vii*) and definition 3.3, we have X_3^* is an optimal solution of (FLP), so $X_3^* \in E_{FLP}$.

* Case 2: $X_{FLP} \in E_{FLP}$. Using theorem 3.1 and lemma 3.2 $E_{FLP} \subseteq E_1$, we also have $X_{FLP} \in E_1$, so $CX_{FLP} = z_1^* = CX_3^*$. Now $X_{FLP} \in E_1 = S_2$ then X_{FLP} is a feasible solution of (II). We continue to divide into two cases when X_{FLP} is or isn't an optimal solution of (II).

.Case 2.1: If $X_{FLP} \in E_2 \Rightarrow \Omega X_{FLP} = z_2^*$. By lemma 3.2 $X_{FLP} \in E_2 = S_3$, so X_{FLP} is a feasible solution of (III). Hence, $\Phi X_{FLP} \leq \Phi X_3^*$ and

$\tilde{z}_{X_{FLP}} = (CX_{FLP}; \Phi X_{FLP}, \Omega X_{FLP})$

$= (z_1^*; \Phi X_{FLP}, z_2^*) \underset{\Re}{\leq} ((z_1^*; \Phi X_3^*, z_2^*) = \tilde{z}_{X_3^*}$

Because X_3^* is a feasible solution of (FLP) (by *vii*), from the above inequality, we have X_3^* is an optimal value of (FLP), that means $X_3^* \in E_{FLP}$.

.Case 2.2: If $X_{FLP} \notin E_2 \Rightarrow \Omega X_{FLP} < z_2^*$. By the definition about $\Re$,we have $\tilde{z}_{X_{FLP}} < \tilde{z}_{X_3^*}$ Since, by (*vii*) X_3^* is a feasible solution of (FLP), so X_3^* is an optimal solution of (FLP), that means $X_3^* \in E_{FLP}$ and proof is completed.

By theorems 3.1 and 3.2, we build the following algorithm to solve the problem (FLP):

Input: $A, B, \tilde{C}$

Output: E_{FLP} , z_{FLP}^*

Algorithm

Step1: Solving the problem (I) to find out the set of optimal solution E_1

- If $E_1 = \varnothing$ then $E_{FLP} = \varnothing$ and jump to step 4.
- If $E_1 \neq \varnothing$ then saving the optimal value z_1^*

Step 2: Solving the problem (II) to find out the set of optimal solution E_2

- If $E_2 = \varnothing$ then $E_{FLP} = \varnothing$ and jump to step 4.
- If $E_2 \neq \varnothing$ then saving the optimal value z_2^*

Step 3: Solving the problem (III) to find out the set of optimal solution E_3

- If $E_3 = \varnothing$ then $E_{FLP} = \varnothing$ and jump to step 4.
- If $E_3 \neq \varnothing$ then saving the optimal value z_3^* and $E_{FLP} = E_3$.

Step 4:

- If $E_{FLP} = \varnothing$ then conclude (FLP) has no optimal solution.
- Else: $E_{FLP} = E_3$ and $\tilde{z}^* = (z_1^*; z_3^*, z_2^*)$

Remark 3.3:

- The above algorithm can be finished at step 1 as soon as we determine that the problem (I) has only one optimal solution: $E_1 = X_1^*$ and X_1^* is the optimal solution of (FLP)

- Similarly, when the problem (II) has only one optimal solution then we can finish the above algorithm at step 2.

Remark 3.4: The above proofs for the case minimization analogues.

4. Illustrative examples

Example 4.1: Production Planning

The Quality Furniture Corporation produces benches and picnic tables. The firm has two main resources: its labor force and a supply of redwood for use in the furniture. During the next production period, 1200 labor hours are available under a union agreement. The firm also has a stock of 5000 pounds of quality redwood. Each bench that Quality Furniture produces requires 4 labor hours and 10 pounds of redwood; each picnic table takes 7 labor hours and 35 pounds of redwood.

The profit of each completed product is predicted in 3 situations: Most Optimistic (MO), Most Likely (ML) and Most Pesimistic (MP) and shown below. How many of benches and tables should be produced to maximize the total profit?

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	Profit ($)		
	MP	ML	MO
Bench	6	9	11
Table	16	20	22

Let x_1 be the number of benches and x_2 is the number of tables to produce. We use triangular fuzzy numbers to represent the profit of each product where ML, MP, MO are core, lower limit and upper limit, respectively. For example, the profit of bench ($\tilde{C}_1$) in ML, MP, MO is 9, 6, 11 then core $(\tilde{C}_1)$ =9; lower limit $(\tilde{C}_1)$ =6, upper limit $(\tilde{C}_1)$ =11 so left spread $(\tilde{C}_1)$ = 3 and right spread $(\tilde{C}_1)$ =2. Thus $\tilde{C}_1 = (9;3,2)$.

Then we have the following problem:

Maximize Profit $\tilde{z} = (9;3,2)x_1 + (20;4,2)x_2$
subject to:
Labor: $4x_1 + 7x_2 \le 1200$ *hours*
Wood: $10x_1 + 35x_2 \le 5000$ *pounds*
$x_1, x_2 \ge 0$, interger

- **Step 1**: Consider the core problem (problem (I)):

$$Max \quad z_1 = 9x_1 + 20x_2$$
$$s.t. \begin{cases} 4x_1 + 7x_2 \le 1200 \\ 10x_1 + 35x_2 \le 5000 \\ x_1, x_2 \ge 0, \quad \text{interger} \end{cases}$$

Using the Excel Solver, we find $z_1^* = 3180$

- **Step 2**: Consider the right spread problem (problem (II)):

$$Max \quad z_2 = 2x_1 + 2x_2$$
$$s.t. \begin{cases} 4x_1 + 7x_2 \le 1200 \\ 10x_1 + 35x_2 \le 5000 \\ x_1, x_2 \ge 0, \quad \text{integer} \\ 9x_1 + 20x_2 = 3180 \end{cases}$$

Solving the above problem, we get $z_2^* = 428$.

- **Step 3**: Now we consider left spread problem:

$$Max \quad z_2 = 3x_1 + 4x_2$$
$$s.t. \begin{cases} 4x_1 + 7x_2 \le 1200 \\ 10x_1 + 35x_2 \le 5000 \\ x_1, x_2 \ge 0, \quad \text{integer} \\ 9x_1 + 20x_2 = 3180 \\ 2x_1 + 2x_2 = 428 \end{cases}$$

and we have $z_3^* = 756$

- **Step 4**:
  + Optimal value: (3180;756,428)
  + Optimal solution: $x_1 = 100, x_2 = 114$

Example 4.2: Blending problem

Consider the example of a manufacturer of animal feed who is producing feed mix for dairy cattle. In our simple example, the feed mix contains two active ingredients and a filler to provide bulk. One kg of feed mix must contain a minimum quantity of each of four nutrients as below:

Nutrient	A	B	C	D
gram	90	50	20	2

The ingredients have the following nutrient values and cost, where cost is estimated in 3 situations: MO, ML, MP:

Ingredient					Cost/kg		
gram/kg)	A	B	C	D	MO	ML	MP
1	100	80	40	10	35	40	50
2	200	150	20	-	50	60	65

What should be the amounts of active ingredients and filler in one kg of feed mix?

In order to solve this problem it is best to think in terms of one kilogram of feed mix. That kilogram is made up of three parts - ingredient 1, ingredient 2 and filler so let:

x_1 = amount (kg) of ingredient 1 in one kg of feed mix
x_2 = amount (kg) of ingredient 2 in one kg of feed mix
x_3 = amount (kg) of filler in one kg of feed mix
where $x_1, x_2, x_3 \ge 0$

As shown above, we use triangular fuzzy number to represent the cost of each ingredient, so this problem is formulated to the following FLP:

$$Min \quad \tilde{z}_{FLP} = (40;5,10)x_1 + (60;10,5)x_2$$
$$s.t. \begin{cases} 100x_1 + 200x_2 \ge 90 \\ 80x_1 + 150x_2 \ge 50 \\ 40x_1 + 20x_2 \ge 20 \\ 10x_1 \ge 2 \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 > 0 \end{cases}$$

- **Step 1**: Solve the core problem:

$$Min \quad z_1 = 40x_1 + 60x_2$$

$$s.t. \begin{cases} 100x_1 + 200x_2 \geq 90 \\ 80x_1 + 150x_2 \geq 50 \\ 40x_1 + 20x_2 \geq 20 \\ 10x_1 \geq 2 \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 > 0 \end{cases}$$

we have $z_1^* = 30.667$

- **Step 2**: Solve the right spread problem:

$$Min \quad z_1 = 10x_1 + 5x_2$$

$$s.t. \begin{cases} 100x_1 + 200x_2 \geq 90 \\ 80x_1 + 150x_2 \geq 50 \\ 40x_1 + 20x_2 \geq 20 \\ 10x_1 \geq 2 \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 > 0 \\ 40x_1 + 60x_2 = 30.667 \end{cases}$$

with an additional constraint: $40x_1 + 60x_2 = 30.667$ and the objective function: $Min\ z_1 = 10x_1 + 5x_2$ we find $z_2^* = 5$

- **Step 3**: Solve the left spread problem:

$$Min \quad z_1 = 5x_1 + 10x_2$$

$$s.t. \begin{cases} 100x_1 + 200x_2 \geq 90 \\ 80x_1 + 150x_2 \geq 50 \\ 40x_1 + 20x_2 \geq 20 \\ 10x_1 \geq 2 \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 > 0 \\ 40x_1 + 60x_2 = 30.667 \\ 10x_1 + 5x_2 = 5 \end{cases}$$

we have $z_3^* = 5.167$

- **Step 4**:
  + Optimal value: (30.667; 5.167,5)
  + Optimal solution: $x_1 = 0.322, x_2 = 0.356$

5. Conclusion

In this article, we present a new definition about comparative relation on the set of fuzzy numbers. By a suitable comparative relation, we build an algorithm to solve the linear programming problem where the objective function is represented by triangular fuzzy numbers. The algorithm has been proved by related theorems and lemmas.

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