Numerical Approach for Local Bifurcation Analysis of Nonlinear Physical System

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Abstract :
In this paper, we propose an study that combines classical linearization method with the Routh Herwitz criterion theory of complex nonlinear systems to compute local stability boundaries and visualize such bifurcation surfaces of nonlinear dynamical systems as function of parameters set (Analytical Search for Bifurcation Surfaces in Parameter Space).
Therefore, we proposed a numerical method for the bifurcation analyses , Our goal is to applied the optimal derivative (based on the minimization in the least-square sense) as introduced by O. Arino and T. Benouaz. In order to gain some progress with this procedure in the term of bifurcation analysis (detection of the local bifurcation in the neighborhood of the bifurcation parameters with respect to an initial condition) . This application enables us to compare the results obtained with those found by the classical linearization (Fréchet derivative (jaccobian matrix) in the equilibriums points) .
Keywords :*Nonlinear ordinary differential equation- optimal derivative- Classical linearization (Freshet derivative in the equilibrium point) - asymptotic stability - bifurcation analysis.*

1. Introduction

Nonlinear models arise for most systems and their presence in one form or another is generally the rule. The source of nonlinearity in structural and mechanical systems may a result of interaction with surrounding, forces can arise due to interacting media or fields. The resulting nonlinear models exhibit a rich variety of phenomena of interest to scientists and engineer. The presence of a bifurcation is of great importance in many physical, chemical and biological systems [1-2].

In the study of nonlinear ordinary differential equations, the linearization method plays an important rôle. In [3–7], Arino and Benouaz have introduced an alternative method termed as the optimal derivative method (see also [8–10]). This is an approximation procedure based on the minimization of a certain functional with respect to a curve starting from an initial value x_0 and going to zero as t goes to infinity.

The localization of critical parameter sets called bifurcations point is often a central task of the analysis of a nonlinear dynamical system. Bifurcations of codimension 1 that can be directly observed in nature and experiments form surfaces in three-dimensional parameter spaces. In this paper, we propose an application of the optimal derivative as introduced by O. Arino and T. Benouaz enables us to compare the results obtained with those found by the classical linearization (Fréchet derivative in the equilibrium point).

in the 1960s and 1970s the mathematical theory of dynamical systems experienced much development, with the introduction of new ideas by Smale, Arnol′d, Lorenz, Yorke, and Feigenbaum, to name just a few of the contributors [8, 9, 10]. With these theoretical developments there came a renewed interest in the dynamics of electronic circuits in the early 1980s, when new ideas and methods were introduced to the study of (periodic or non-periodic) oscillations generated by nonlinear electronic circuits of low dimension. However, the theory usually only provides a framework for different phenomena that one may find in a given circuit. To perform an effective study of the actual dynamics it is necessary to resort to numerical methods. In order to obtain a global view of the dynamics in phase space and of the bifurcations in parameter space, one needs to employ numerical methods that go beyond mere numerical simulation.

In this paper we demonstrate how complicated dynamical behavior and bifurcations can be found and identify in ODE models of physical system. The combination of theoretical methods (Routh Hurwitz criterion and linear algebra) and numerical technique (optimal derivative) allows one to obtain a deep understanding of a wide range of dynamical phenomena. In particular, physical system provide concrete examples of unfolding of singularities that act as organizing centers of the dynamics.
Our intention is to apply and make some progress with this procedure in the term of bifurcation analysis (detection of the local bifurcation in the vicinity of the bifurcation parameters with respect to an initial condition) .

2. Optimal Derivative Review

Consider a nonlinear ordinary differential problem of the form (see [3–7])

$$\begin{cases} \dfrac{dx}{dt} = F(x(t)) \\ x(0) = x_0 \end{cases}$$

where

- $x = (x_1, \cdots, x_n)$ is the unknown function,
- $F = (f_1, \cdots, f_n)$ is a given function on a open subset Ω of IR^n,

with the assumptions:
$H1$) $F(0,0) = 0$.
$H2$) F is γ Lipchitz continuous,
$H3$) The spectrum $\sigma(DF(x))$ is contained in the set $\{z : \mathrm{Re}\, z < 0\}$ for every $x \neq 0$, in a neighborhood of 0, for which $DF(x)$ exist.

Given $x_0 \in IR^n$, we choose a first linear map A_0. For example, if F is differentiable in x_0, then we can take $A_0 = DF(x_0)$ or the derivative value in a point in the vicinity of x_0 . This is always possible if F is locally Lipschitz. Now, let y_0 be the solution of the initial value problem

$$\dot{y} = A_0\, y(t), \quad y(0) = x_0 \tag{1}$$

Next, we minimize the functional

$$G(A) = \int_0^{+\infty} \left\| F(y_0(t)) - A\, y_0(t) \right\|^2 dt. \tag{2}$$

This minimization problem is always uniquely solvable, and as the optimal linear map minimizing (2) we obtain

$$A_1 = \left(\int_0^{+\infty} [F(y_0(t))][y_0(t)]^T dt \right) \times \left(\int_0^{+\infty} [y_0(t)][y_0(t)]^T dt \right)^{-1}. \tag{3}$$

Now we define y_1 to be the solution of Eq(1) with A_0 replaced by A_1 and we minimize Eq(2) with y_0 replaced by y_1. Then we continue in this way. The optimal derivative $\widetilde{A}$ is the limit of the sequence build as such (for details, see [4–8]).

3. Presentation of the illustrative example:

The example derived from a Nonlinear mechanical system representing a forced nonlinear oscillator [11],[12] , in fact is a mechanical posistioning device with feed-back control. given by the system:

$$\ddot{x} + \delta\dot{x} + K(x)x = -z + F(t)$$
$$\dot{z} + \alpha z = \alpha\gamma(x - r)$$

x is defined as the displacement, $\delta\dot{x}$ the linear damping with a damping constant $\delta > 0$, object of negative feed-back control (z) with time constant $1/\alpha$ and the gain γ) , $K(x) = (x^2 - 1)$:

We take $F(t) = 0$, $r = 0$ representing autonomous system, in which the governing equation have no explicit time dependence. Our goal that this type of dynamical systems can still exhibit complicated dynamics (complex bifurcations and transient to chaos) with a regime in with two or more stables limit cycles exist :

$$\begin{aligned} &\ddot{x} + \delta\dot{x} + x^3 - x = -z \\ &\dot{z} + \alpha z = \alpha\gamma x \end{aligned} \tag{4}$$

the system (4) with the dimensionless equation is given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 - \delta y - z \\ \dot{z} = \gamma\alpha x - \alpha z \end{cases} \tag{5}$$

$(x, y, z) \in IR^3$ with the parameters $\delta, \alpha, \gamma > 0$.

4. Analytical Search for Bifurcation surfaces in Parameter Space

To locate the bifurcations of a given system in parameter space is one of the main tasks of qualitative analysis. local bifurcations correspond to qualitative changes in the neighborhood of one steady state and can, therefore, be detected by monitoring the eigenvalues of the corresponding Jacobian matrix.

4.1 Determination of the equilibrium points

we can see that there exists an equilibrium in IR^3 if and only if the equations

$$\begin{cases} \dot{x} = y = 0 \\ \dot{y} = x - x^3 - \delta y - z = 0 \\ \dot{z} = \gamma\alpha x - \alpha z = 0 \end{cases} \begin{cases} x = 0 \\ x = \pm\sqrt{1-\gamma} \\ z = \gamma\sqrt{1-\gamma} \end{cases} \quad (6)$$

The system has one trivial steady state with the origin $(x, y, z) = (0,0,0)$ and two symmetric non-trivial steady states (P_+, P_-) with $(x, y, z) = \left(\pm\sqrt{1-\gamma}, 0, \pm\gamma\sqrt{1-\gamma}\right)$. In all steady states $x = 0$ and $z = x(1 - x^2)$ hold. We can conclude the following results :

- Case 1: if $\gamma \geq 1$ the system has one trivial steady state at the origin with $x = 0$ (for any values of the parameters). In fact for $\gamma = 1$, the three equilibria coincide, and we therefore have pitchfork bifurcation at $\gamma = 1$.
- Case 2 : two steady states appear if $0 < \gamma < 1$.

The stability of equilibrium states can be determined by evaluating the Jacobian matrix at this states and then examining its eigenvalues (for the case 1 and 2).

4.2 Behaviour of the system around O(0,0,0) $\gamma \geq 1$

Jacobian matrix DF(x) at the steady state $\boldsymbol{x = 0}$ is given by (7)

$$J(0) = DF(0) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\partial g}{\partial x}(0) & -\delta & -1 \\ \alpha\gamma & 0 & -\alpha \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\delta & -1 \\ \alpha\gamma & 0 & -\alpha \end{bmatrix} \quad (27)$$

with $\frac{\partial g}{\partial x}(x) = 1 - 3x^2$

By $det(\lambda I - J) = 0$, we get the characteristic equation corresponding to (7)

$$P(\lambda) = \lambda^3 + (\alpha + \delta)\lambda^2 - (1 - \alpha\delta)\lambda - \alpha(1 - \gamma) = 0 \quad (8)$$

In order to compute local stability boundaries for which the equilibrium exchange stability, then we detect such bifurcation surfaces of nonlinear dynamical systems as function of system parameters δ, α, γ , we use the Routh-Herwitz criterion theory [10],[11].

we note that in this range of $\gamma \geq 1$, we have two stability boundaries function of the parameters set,

the first one is $\gamma = 1$. the second can be determined by applying Routh–Hurwitz criterion as follow :

From the coefficients of the polynomial characteristic equation (8) we can construct the Routh matrix given by :

$$H = \begin{bmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ with } a_0 = 1, a_1 = (\alpha + \delta), a_2 = -(1 - \alpha\delta), a_3 = -\alpha(1 - \gamma).$$

The polynomial P is asymptotically uniformemlly stable if $a_0 > 0, a_2 > 0, a_1 a_2 > a_3$ and the principal determinants of the matrix H are strictly positive i.e $\Delta_1 = a_1 > 0, \Delta_2 = a_1 a_2 - a_3 a_0 > 0$

we imply that for $\gamma > 1$ and $\alpha\delta > 1$, the second local stability boundary is given by

$$\boldsymbol{\gamma = \gamma_H} = \frac{\delta}{\alpha}(\alpha^2 + \alpha\delta - 1) \quad (9)$$

using this boundary values characterized by a critical value of the bifurcation parameter, we can compute the eigenvalues of the associated linear system function of the parameters set δ, α, γ.

Note: *The advantage of this criterion that we can analyzed the stability of a system by locating the stability boundaries without information on eigenvectors of the equivalent linear system.*

- case where $\gamma = 1$

the characteristic equation (7) became :

$$\lambda^3 + (\alpha + \beta)\lambda^2 - (1 - \alpha\delta)\lambda = 0. \quad (10)$$

the roots of this equation are given by:

$$\begin{cases} \lambda_1 = 0 \\ \lambda_{2,3} = -\frac{1}{2}(\alpha + \delta) \pm \frac{1}{2}\sqrt{(\alpha - \delta)^2 + 4} \end{cases} \quad (11)$$

Note that the term $(\alpha - \delta)^2 + 4$ is always positive, therefore the eigenvalues $\lambda_{2,3}$ are real and one become zero $\lambda_1 = 0$.

The origin is always an equilibrium. A pitchfork bifurcation of equilibria occurs on the plane, which creates two symmetry-related equilibria. This situation is characterized by the bifurcation surface $\boldsymbol{\gamma = 1}$ (see Fig 2 later).

- Case where $\gamma = \gamma_H = \frac{\delta}{\alpha}(\alpha^2 + \alpha\delta - 1)$ for ($\gamma > 1$ and $\alpha\delta > 1$)

the equation (8) become :

$$P(\lambda) = \lambda^3 + (\alpha + \delta)\lambda^2 - (1 - \alpha\delta)\lambda - \alpha\left(1 - \frac{\delta}{\alpha}(\alpha^2 + \alpha\delta - 1)\right) = 0$$

the roots of this equation are given by:

$$\begin{cases} \lambda_1 = -\alpha - \delta \\ \lambda_{2,3} = \pm\sqrt{(1 - \alpha\delta)} \end{cases} \quad (12)$$

for $\alpha\delta > 1$ we have : $(1 - \alpha\delta) < 0$. We can rewrite Eq (12) in the following form

$$\begin{cases} \lambda_1 = -\alpha - \delta \\ \lambda_{2,3} = 0.0 \pm i\sqrt{(1 - \alpha\delta)} \end{cases}$$

In this case the jacobian matrix have a pair of complex conjugate eigenvalues crosses the imaginary axis. If the system was in a stable steady state before the bifurcation, the steady state loses it's stability at the bifurcation point. Furthermore, a stable or unstable limit cycle emerges or vanishes. This type of elementary bifurcation is called the *Complex Hopf bifurcation*.

we can conclude that there exists a Hopf bifurcation emerging from its equilibrium at origin when $\boldsymbol{\gamma}$, passes through the critical value $\gamma = \gamma_H > 1$. and *the surface* described by Eq. (9) is a *Hopf bifurcation (HB) surface.*

Furthermore, when the delay τ>τ0, the positive equilibrium at origin loses its stability and the system goes into oscillations. The system undergoes a Hopf bifurcation at $\gamma = \gamma_H > 1$.

4.3 Behaviour of the system around two equilibrium points (P_+, P_-)

The equations of the system Eq (5) are invariant to the transformation $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{x} - \boldsymbol{x}, -\boldsymbol{y}, -\boldsymbol{z})$

so we note that due to this symmetry any asymmetric limit set coexist with another one which topologically similar.

In this range $0 < \gamma < 1$, the fixed point at the origin is no longer stable, and two additional fixed points (P_+, P_-) appear in the phase space of the nonlinear system. The coordinates of these fixed points are given by $(x, y, z) = \left(\pm\sqrt{1-\gamma}, 0, \pm\gamma\sqrt{1-\gamma}\right)$. The Jacobian matrix of system Eq (5) for (P_+, P_-) is given by :

$$DF(P_\pm) = \begin{bmatrix} 0 & 1 & 0 \\ 1 - 3\left(\pm\sqrt{1-\gamma}\right)^2 & -\delta & -1 \\ \alpha\gamma & 0 & -\alpha \end{bmatrix} \tag{13}$$

by $det(\lambda I - J) = 0$, we get the characteristic equation corresponding to Eq (13)

$$P(\lambda) = \lambda^3 + (\alpha + \delta)\lambda^2 - (\alpha\delta + 2 - 3\gamma)\lambda - 2\alpha(1 - \gamma) = 0. \tag{14}$$

Applying Routh–Hurwitz criterion using the coefficients of the equation (14) , we can compute a new stability boundary $(\boldsymbol{\gamma_H})$ in the range $0 < \gamma < 1$ given by:

$$\gamma = \gamma_{H1} = \frac{\delta}{\alpha + 3\delta}\left(\alpha^2 + \alpha\delta + 2\right) \text{ with } \alpha\delta < 1 \tag{15}$$

an new bifurcation surface characterized by the equation (15) have been found. an examination of the eigenvalues reveal later that is *Hopf bifurcation surface.*

Replacing γ by the equation (15) in the jacobian matrix given by (13) we obtain :

$$\begin{cases} \lambda_1 = -\alpha - \delta \\ \lambda_{2,3} = \pm\dfrac{1}{-\alpha - 3\delta}\sqrt{2}(\alpha + \delta)\sqrt{-\left((\alpha + 3\delta)\alpha(\alpha\delta - 1)\right)} \end{cases}$$

for $\alpha\delta < 1$, we have

$$\begin{cases} \lambda_1 = -\alpha - \delta \\ \lambda_{2,3} = 0.00 \pm \left[\frac{1}{-\alpha-3\delta}\sqrt{2}(\alpha + \delta)\sqrt{-\left((\alpha + 3\delta)\alpha(\alpha\delta - 1)\right)}\right] i \end{cases} \tag{16}$$

In this case the jacobian matrix have a pair of complex conjugate eigenvalues Eq (16) crosses the imaginary axis. hence the nonlinear system undergoes a other Hopf bifurcation.

While as $\boldsymbol{\gamma}$ is increased to pass $\boldsymbol{\gamma_H}$, Hopf bifurcation occurs, i.e. a family of periodic solutions bifurcate (bifurcating periodic solutions) , small values of the δ the damping constant $(\alpha\delta < 1)$.

An analytical approach yielding the bifurcation as a function of all system parameters. this reveals The analysis reveals a Hopf bifurcation surface (Fig. 1); characterized by the equations $\boldsymbol{\gamma = 1}, \boldsymbol{\gamma = \gamma_{H_1}} = \frac{\delta}{\alpha}(\alpha^2 + \alpha\delta - 1)$ with $\alpha\delta > 1$ and

$\gamma = \gamma_{H1} = \dfrac{\delta}{\alpha + 3\delta}\left(\alpha^2 + \alpha\delta + 2\right)$ with $\alpha\delta < 1$

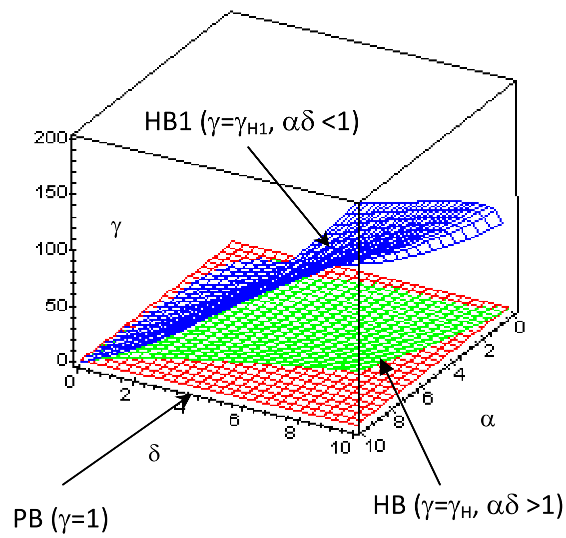


Fig 1: The Hopf bifurcation surface (HB) (green grey) and the Hopf bifurcation (HB1) (blue grey) surface of PB (red grey)(pitchfork bifurcation) depending on the three parameters $(\alpha, \delta, \gamma > 0)$,

The two green and blue surfaces correspond to Hopf bifurcations while the red surface will in general be a pitchfork bifurcation these bifurcation occur if an eigenvalue of the Jacobian becomes zero (see Eq (11)). A double Hopf (DH) bifurcation line is formed at the intersection of the two Hopf surface surfaces. we can see that the line of this intersection is characterized by the following condition:

$\gamma = \gamma_H = \gamma_{H_1} = 1$

This yield $\begin{cases} \gamma = \gamma_H = \frac{\delta}{\alpha}\left(\alpha^2 + \alpha\delta - 1\right) = 1 \Rightarrow \alpha\delta = 1 \\ \gamma = \gamma_{H1} = \frac{\delta}{\alpha + 3\delta}(\alpha^2 + \alpha\delta + 2) = 1 \Rightarrow \alpha\delta = 1. \end{cases}$

we obtain $\gamma = 1$ and $\delta = \frac{1}{\alpha}$, this line is visible in the following figure

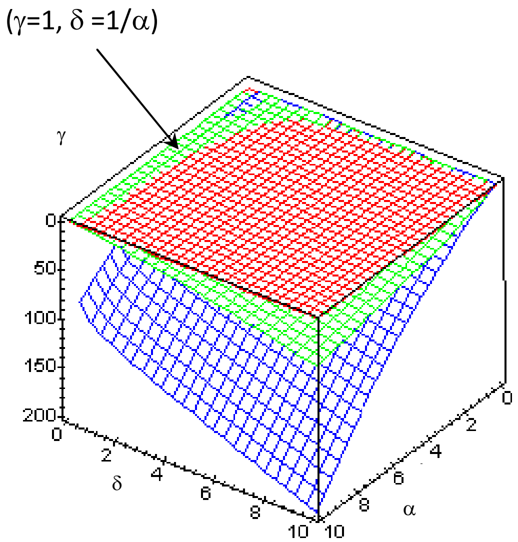


Fig 2: the inversion of the fig 2 (the view of the intersection of the three bifurcation surfaces)

using the condition of intersection we can summarize that:

- $\gamma > 1$, There is an equilibrium at the origin O(0,0,0), for any values of parameters. A Hopf bifurcation occur characterized by HB ($\boldsymbol{\gamma_H}$) for the choose of the parameter (α,δ) such $\alpha\delta > 1$ i.e (above the line $\gamma = 1$ and $\delta = \frac{1}{\alpha}$).

- There is two equilibrium points $(\boldsymbol{P_+}, \boldsymbol{P_-})$. A Hopf bifurcation occur characterized by HB (γ_{H_1}) for the choose of the parameter (α,δ) such $\alpha\delta > 1$ i.e (below the line $\gamma = 1$ and $\delta = \frac{1}{\alpha}$).

At the Hopf bifurcation point the involved equilibrium becomes unstable when it was

stable (supercritical) or stable. . Also, a periodic solution, a limit cycle, is born that inherits the stability properties that the equilibrium had before the occurrence of the bifurcation.

5. Numerical analysis of system using the optimal derivative

In this section, we perform numerical simulation of system Eq (5) using the optimal derivative cited in section 2. the stability on either side of the surface can be found by running a simulation or calculating the eigenvalues of the optimal linear matrix numerically at a points in the neighborhood of the equilibrium points on each side of the surfaces.

5.1 Linearization around equilibrium state O (0, 0, 0) for $\gamma \geq 1$:

5.1.1 Linearization around equilibrium state $O(0,0,0)$ for $\gamma \geq 1$

In this range there is a critical value of the bifurcation parameter $\gamma_H = \frac{\delta}{\alpha}\left(\alpha^2 + \alpha\delta - 1\right)$, we distinguish two range of parameter variation delimited by γ_H as follow:

$$\begin{cases} 1 < \gamma < \gamma_H \text{ (the equilibrium is asymptotically stable)} \\ \gamma = \gamma_H \text{ (The system undergoes a Hopf bifurcation)} \\ \gamma > \gamma_H > 1 \text{ (the equilibrium point is instable).} \end{cases} \quad (17)$$

We Carry out a numerical simulations using the optimal derivation in this ranges . For our application we choose the values of the parameter $\alpha, \delta > 0$,we take $\alpha = 3;\ \delta = 2$ with $\alpha\delta > 1$.

First we investigate :

The case $\gamma = 1$: We choose an initial condition near the origin $(x_0, y_0, z_0) = (0.001, 0.02, 0.04)$

The optimal derivative procedure gives (with the accuracy $\varepsilon = 10^{-6}$) :

$$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1.000 & -1.9999 & -1 \\ 3 & 0 & -3 \end{bmatrix}, \quad (18)$$

which has the eigenvalues

$$\begin{cases} \lambda_1 = -1.537 \cdot 10^{-6} \\ \lambda_2 = -1.3819 \\ \lambda_3 = -3.6180\,. \end{cases} \quad (19)$$

Figures 3 show, the phase space of the optimal linear system (18), compared to the nonlinear system Eq (5)

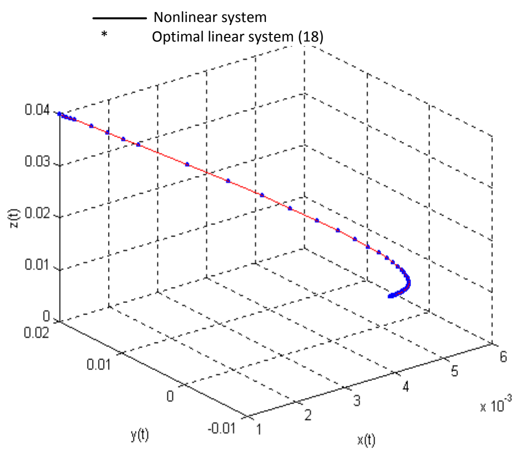


Fig 3: The phase space of (18) and (5) when $(x_0, y_0, z_0) = (0.001, 0.02, 0.04)$

one of the eigenvalues of the the jacobian matrix becomes zero Eq (11). The origin is always an equilibrium. A pitchfork bifurcation of equilibria occurs, which creates two symmetry-related equilibria. This situation is characterized by the bifurcation surface $\boldsymbol{\gamma = 1}$. The solution of optimal linear system and nonlinear one are identical (see figure 3) same dynamical behavior . In term of eigenvalues ($\lambda_1 \approx O(10^{-6})$) is very close to zero,thus indicating the same conclusion .

The case $1 < \gamma \leq \gamma_H$

The origin is always the equilibrium point in this range. We take $\gamma = 2$,

we choose the values of the parameter $\alpha, \delta > 0$, we take $\alpha = 3;\ \delta = 2$ with $\alpha\delta > 1$ and initial value $(x_0, y_0, z_0) = (0.1, 0.2, 0.4)$. The optimal derivative procedure gives (with $\varepsilon = 10^{-6}$) :

$$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0.96 & -2.000 & -1.000 \\ 6.000 & 0 & -3.000 \end{bmatrix}, \qquad (20)$$

which has the eigenvalues

$$\begin{cases} \lambda_1 = -3.91329 \\ \lambda_{2,3} = -0.54599 \pm i\ 0.69505 \end{cases} \qquad (21)$$

To illustrate graphically the results obtained above, we have plotted in Figure 4 the solution x(t) verus of the nonlinear system (5) compared to the solution given by the optimal linear system (20). Figure 5, show the phase space of the optimal linear system (20), compared to the nonlinear system (5), for the initial conditions (x0, y0,z0)= (0.1, 0.2, 0.4).

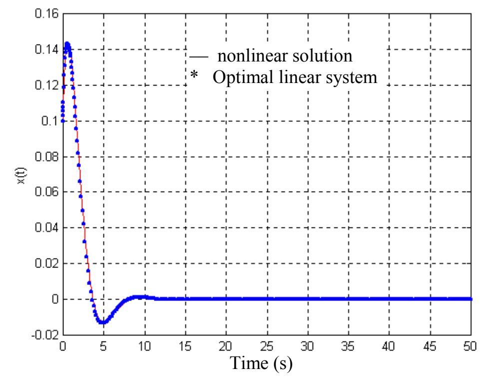


Fig 4 : The variation of x as a function of time when (x0, y0,z0)= (0.1, 0.2, 0.4)

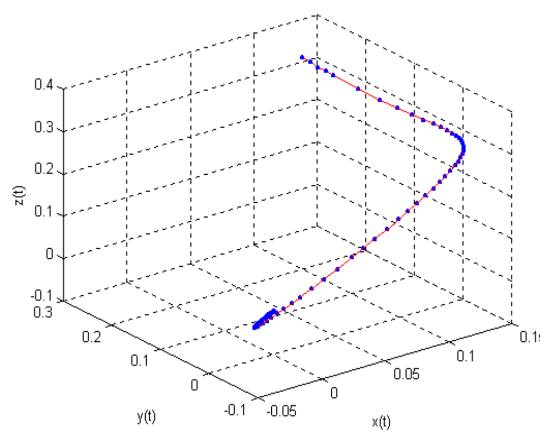


Figure 5: The solution (x(t),y(t),z(t)) in the phase space.

We examine the stability near this equilibrium point, The real parts of both eigenvalues λ2,3 (determine the stability) are negative explain the damped oscillation, λ1 real and strictly negative determine the attractive comportment (fast dynamic) . Thus the optimal linearization is asymptotically stable and shows the origin as a focus (Fig 5). Therefore the origin is asymptotically stable (Fig 4). therefore the qualitative analysis show that the optimal linear system is identical-equivalent- (describe the same dynamical behaviour) and lead to the same conclusion as the classical linearization, showing the origin as a focus, hence being asymptotically stable.

The case $\gamma = \gamma_H = \frac{\delta}{\alpha}\left(\alpha^2 + \alpha\delta - 1\right)$ with $\alpha\delta > 1$:

the jacobian matrix has a pair of complex eigeinvalues with purely imaginer part a Hopf bifurcation occur. We applied the optimal derivative in order to detect this bifurcation. For the same parameter values

$\alpha = 3, \delta = 2, \gamma_H = 9.3333 \text{ et } \alpha\delta > 1,$ and initial value $(x_0, y_0, z_0) = (0.01, 0.02, 0.04)$. The optimal derivative procedure after 2 iterations gives (with $\varepsilon = 10^{-6}$) :

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0.99988 & -2 & -1 \\ 28 & 0 & -3 \end{bmatrix}, \qquad (22)$$

which has the eigenvalues $\begin{cases} \lambda_1 = -4.9999 \\ \lambda_{2,3} = -2.08 \cdot 10^{-5} \pm 2.2360i \end{cases}$

The eigenvalues is approching zero as we approche the bifurcation parameter and for initial condition condition very close to zero $\mathrm{Re}(\lambda_{2,3}) \approx O(10^{-5})$.

we have plotted in Figure 6 the solutions x (t) versus time of the nonlinear system Eq (1) compared to the solution given by the optimal linear system (22). Figure 7, show the phase space of the optimal linear system Eq (22), compared to the nonlinear system Eq(5), for $\gamma = \gamma_H = 9.3333$ and the initial conditions (x0, y0,z0) = (0.01, 0.02, 0.04).

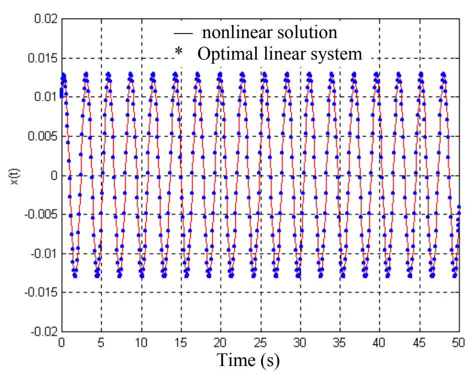


Fig 6 : The variation of x as a function of time when $\gamma = \gamma_H = 9.3333$

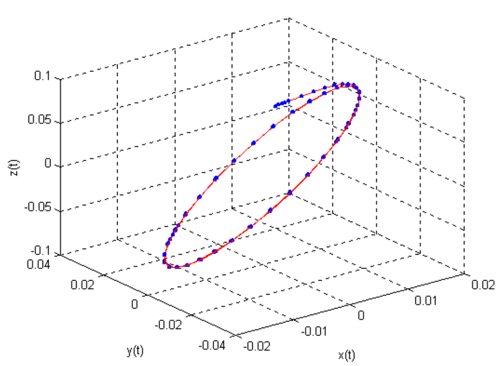


Figure 7: The the phase space for $\gamma = \gamma_H = 9.3333$

the linear optimal system for an initial condition near the origin, appearance of oscillatory behavior in the system as γ increases from 1 (see Fig 6), therefore we have shown in Fig 7 that a limit cycle appears at the Hopf bifurcation when crossing the parameter boundary $\gamma = \gamma_H = 9.3333$.

This bifurcation involved equilibrium becomes unstable when it was stable (supercritical) or stable when it was unstable (subcritical). Also, a periodic solution, a limit cycle, is born that inherits the stability properties that the equilibrium had before the occurrence of the bifurcation. See Figure 7 for a phase plot where a limit cycle is depicted. The both Figures shows already that we have a good approximation.

The case $\gamma > \gamma_H$: the critical value $\gamma = \gamma_H = 9.3333$, define a stability boundary for this range the unique equilibrium point exchange his stability and become instable. in order to verify this result using the numerical proposed method. We take $\gamma = 10 > \gamma_H = 9.3333$.

The optimal derivative procedure gives (with $\varepsilon = 10^{-6}$)

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1.02 & -2.01 & -1.005 \\ -30 & 0 & -3 \end{bmatrix}, \qquad (23)$$

which has the eigenvalues

$$\begin{cases} \lambda_1 = -5.075825. \\ \lambda_{2,3} = 0.032468 \pm i\, 2.309864 \end{cases}$$

The real parts of both $\lambda_{2,3}$ are positive. Thus the optimal linearization is unstable and shows the origin as a focus. Therefore the origin is instable. So proposed numerical methods can detect this instability.

The Pitchfork Bifurcation at $\gamma = 1$: for $\gamma > 1$, there is one real equilibrium. For $\gamma < 1$ (decreasing **γ**) , there are three the origin and (P_+, P_-). At **γ = 1**, the three equilibria are all at the origin, so we have a *pitchfork bifurcation*. For **γ** near 1, the new equilibria are close to the origin.

5.2 Behaviour of the system around the new equilibria (P_+, P_-)

5.2.1 Linearization around the two equilibrium (P_+, P_-) in the range $0 < \gamma < 1$

To examine the stability using the proposed method (optimal derivative) around this point .We translate this points (P_+, P_-) = $(\pm\sqrt{1-\gamma}, 0, \pm\gamma\sqrt{1-\gamma})$ to the origin in order to satisfy the Assumption $(H1) F(0) = 0$ (see section 2) , using the change of variables $(X, Y, Z) = (x - x_1, y - y_1, z - z_1)$, $(x_1, y_1, z_1) = (\pm\sqrt{1-\gamma}, 0, \pm\gamma\sqrt{1-\gamma})$. This gives the system

$$\begin{cases} \dot{X} = (Y + y_1) \\ \dot{Y} = (X + x_1) - (X + x_1)^3 - \delta(Y + y_1) - (Z + z_1) \\ \dot{Z} = \gamma\alpha(X + x_1) - \alpha(Z + z_1). \end{cases} \quad (24)$$

In this range $0 < \gamma < 1$ the classical linearization show that there is a critical value of the bifurcation parameter $\gamma_{H_1} = \frac{\delta}{\alpha + 3\delta}\left(\alpha^2 + \alpha\delta + 2\right)$, we distinguish two range of parameter variation delimited by γ_{H1} as follow:

$$\begin{cases} 1 < \gamma < \gamma_{H1} \text{ (P}_+\text{, P}_-\text{ are asymptotically stable)} \\ \gamma = \gamma_{H1} \text{ (Hopf bifurcation)} \\ \gamma > \gamma_{H1} > 1 \text{ (P}_+\text{, P}_-\text{ points are instable).} \end{cases} \quad (25)$$

The case $\boldsymbol{\gamma < \gamma_{H_1} < 1}$:

We applied the proposed numerical simulation in this ranges . For our application we choose the values of the parameter $\alpha, \delta > 0$,we take

$\alpha = 0.4,\ \delta = 0.2,$ with $\alpha\delta < 1,$ and $0 < \gamma < 1,$

The optimal derivative procedure gives (with $\varepsilon = 10^{-6}$) :

$$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1.1990 & -0.16834 & -0.78085 \\ 0.27 & 0 & -0.9 \end{bmatrix}, \quad (26)$$

which has the eigenvalues

$$\begin{cases} \lambda_1 = -1.00565 \\ \lambda_{2,3} = -0.032716 \pm i\, 1.134068 \end{cases} \quad (27)$$

Figure 8 the solutions x (t) versus time of the nonlinear system Eq (1) compared to the solution given by the optimal linear system Eq (26). Figure 9, show the phase space of the optimal linear system Eq(26), compared to the nonlinear system Eq (5), for $\gamma = 0.3$ and the initial inditions (x_0, y_0,z_0)= (0.1, 0.2, 0.4).

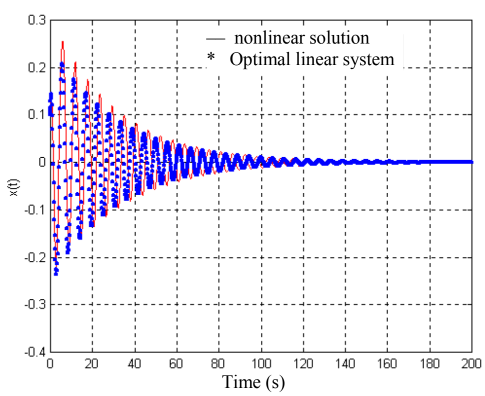
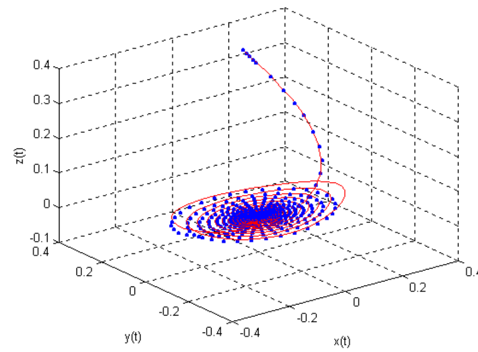


Fig 8 : The variation of x as a function of time when $\gamma = 0.3$



Fige 9: The solution (x(t),y(t),z(t)) in the phase space for $\gamma = 0.3$.

The real parts of both eigenvalues $\lambda_{2,3}$ (determine the stability) are negative. Such terms represent exponentially decaying oscillations (damped oscillation), λ1 real and strictly negative determine the attractive comportment (fast dynamic) . Thus the optimal linearization is asymptotically stable and shows the equilibrium points (P_+,P_-) as a focus stable spiral) (Fig 9). Therefore the new equilibrium points (P_+,P_-) are asymptotically stable (Fig 8). therefore the qualitative analysis show that the optimal linear system is identical-equivalent- (describe the same dynamical behaviour) and lead to the same conclusion as the classical linearization.
the solution leaving the origin spirals into the nearest equilibrium (P_+ or P_-) (makes one loop around this). the optimal linearized system predicts a a spiral, then the fixed points really is a spiral for the original nonlinear system.

The case $\gamma = \gamma_{H_1}$:

the jacobian matrix has a pair of complex eigeinvalues with purely imaginer part a Hopf bifurcation occur. We applied the optimal derivative in order to detect this bifurcation. We take the parameters values:
$\alpha = 0.9, \delta = 0.2,$

hence $\gamma = \gamma_{H_1} = \frac{\delta}{\alpha + 3\delta}(\alpha^2 + \alpha\delta + 2) = 0.3987$ and $\alpha\delta < 1.$

The optimal derivative procedure after 3 iterations gives (with $\varepsilon = 10^{-6}$) :

$$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ -0.80793 & -0.1963 & -0.98185 \\ 0.35879 & 0 & -0.9 \end{bmatrix}, \quad (28)$$

which has the eigenvalues

$$\begin{cases} \lambda_1 = -1.0963 \\ \lambda_{2,3} = 6.7886 \cdot 10^{-5} \pm i\, 0.99251 \end{cases} \quad (29)$$

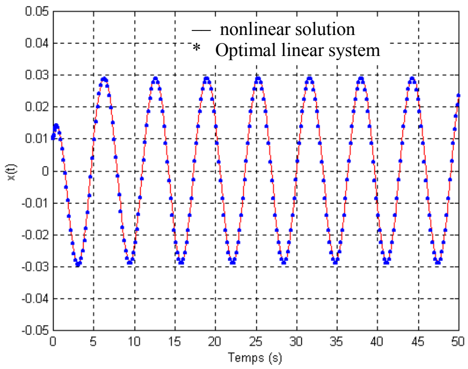


Fig 10: the solutions x (t) versus time of the nonlinear system Eq(5) compared to the solution given by the optimal linear system Eq (28).

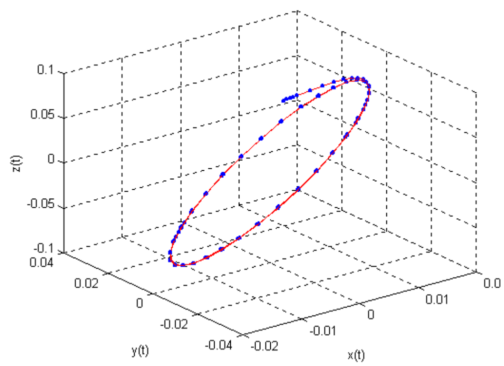


Fig 11: show the phase space of the optimal linear system (28), compared to the nonlinear system (5), for $\gamma = \gamma_{H1} = 0.3987$ and the initial conditions (x_0, y_0,z_0) = (0.01, 0.02, 0.04).

First eigenvalue in Eq (29) is negative because ($\alpha > 0, \delta > 0$), so that the associated eigendirection is attractive and the flow is directed towards the basin of attraction of the equilibrium points .

The above calculations verify that the equilibria (P_+,P_-) are stable spirals until we reach $\gamma = \gamma_{H1} = 0.3987$, at which point they become unstable spirals. Although the calculations of unstable limit cycles are beyond our capabilities here, it can be shown that in fact $\gamma = \gamma_{H1} = 0.3987$ is a subcritical Hopf bifurcation in which, as γ is decreased, the unstable limit cycles are absorbed by the equilibria which then become unstable themselves. so The one large negative real eigenvalues Eq (27) ($\lambda_{2,3} = -0.03271 \pm i\, 1.13406$) tells us that the solutions rapidly approach the plane of the spiral, after which they slowly spiral in (see Fig 11). Let's try to observe at least part of the spiral by integrating forward in time.

The case $\gamma > \gamma_{H_1}$:

Crossing the critical value $\gamma = \gamma_{H1} = 0.3987$, Thus the new equilibrium points (P_+,P_-) goes from stable to unstable as γ increases through γ_{H_1} .

using the numerical proposed method. We take $\gamma = 0.6 > \gamma_{H1} = 0.3987$

The optimal derivative procedure after 2 iterations gives (with $\varepsilon = 10^{-6}$) :

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ -0.29003 & -0.1966 & -0.82834 \\ 0.54 & 0 & -0.9 \end{bmatrix}, \tag{30}$$

which has the eigenvalues

$$\begin{cases} \lambda_1 = -1.19490. \\ \lambda_{2,3} = 0.06261 - i\ 0.76820 \end{cases} \tag{31}$$

The real parts of both $\lambda_{2,3}$ are positive. Thus the optimal linearization is unstable and shows the new equilibrium points (P_+,P_-) instable focus (instable spiral). Therefore the origin is instable. So proposed numerical methods can detect this instability.

We can note, the above calculations verify that the equilibria (P_+,P_-) are stable spirals until we reach $\gamma = \gamma_{H1} = 0.3987$, at which point they become unstable spirals. Although the calculations of unstable limit cycles are beyond our capabilities here, it can be shown that in fact $\gamma = \gamma_{H1} = 0.3987$ is a subcritical Hopf bifurcation in which, as γ is decreased, the unstable limit cycles are absorbed by the equilibria which then become unstable themselves.

6. RESULTS AND DISCUSSION

The stability of the equilibria, and the local bifurcation analysis of the model, are described . We will discuss some of the basic results. We now summarize and compare the results obtained the analytical analysis in order to confirm and validate the results obtained using the optimal derivative.

we can deduce from the determination of the bifurcation surfaces that the higher codimension bifurcations can easily be spotted, once the full parameter dependence of the bifurcation surfaces is known. In the three-parameter bifurcation diagram codimension two bifurcations appear as lines ($\gamma = 1$ and the time constant $\delta = 1/\alpha$) at which codimension one surfaces intersect (see Figure 1 and 2 (HB, HB1) and PB each other.

We now summarize the local bifurcations of the nonlinear system (5) in dependence on the parameters γ. The origin is always an equilibrium. A pitchfork bifurcation of equilibria occurs at γ =1 , which creates two symmetry-related equilibria that exist for $\gamma < \gamma_{H_1} < 1$. The origin as well as the trivial equilibria undergo a Hopf bifurcation in the range $1<\gamma\leq\gamma_H$. The main event in this parameter range is the increasingly oscillatory nature of the solution as increases γ. crossing the critical value $\gamma=\gamma_H$ there exists a small amplitude repelling periodic orbit enclosing a stable equilibrium. The main event in this parameter range is the increasingly oscillatory nature of the solution as increases the gain γ.

For γ > 1, there are three equilibrium points. At γ = 1, the three equilibria are all at the origin, so we have a pitchfork bifurcation.

The next event along the γ -trail is a change in the character of the equilibria equilibria (P_+,P_-) from stable to instable spirals. As shown below, this occurs at about r = 1.346. This is correspond to appearance of oscillatory behavior in the system as γ increases from 1 . The other event in this range is the shrinking of the unstable limit cycles around (P_+,P). The cycles are heading for a Hopf bifurcation which occurs at $\gamma=\gamma_{H1}$.

Finally, The results show good agreement between the theoretical study and numerical analysis carried out from the procedure of optimal derivative (see Figure 6-7-8-9-10-11). the qualitative analysis show that the optimal linear system is identical-equivalent- (describe the same dynamical behaviour) and lead to the same conclusion.

What happens immediately after the hopf bifurcations?

This requires this simple-looking deterministic system could have extremely erratic dynamics: over a wide range of parameters, the solutions oscillate irregularly, never exactly repeating but always remaining in a bounded region of phase space the trajectories in three dimensions, settled onto a complicated set, now called a strange attractor. So hopf bifurcations cascade transient the nonlinear system to chaotic attractor.

7. Conclusions

the well-known criteria for local and bifurcation conditions of equilibria were applied using The Routh–Hurwitz stability criterion and linear system matrix were applied to determine the Equation for the bifurcation surfaces in the parameter space as homogeneous polynomials of the system parameters. Compute numerically and visualize such bifurcation surfaces in a very efficient way. The visualization can enhance the qualitative understanding of a system. Moreover, it can help to quickly locate more complex bifurcation situations corresponding to bifurcations of higher codimension at the intersections of bifurcation surfaces.

an interesting local bifurcation phenomenon is described using the linear optimal system associated with the original nonlinear system . The bifurcation and stability analysis using this new numerical method confirms the results obtained by symbolic analysis using the linear system matrix and the Routh–Hurwitz criteria. It also shows its potential to be a tool for analyzing the stability of this type nonlinear ordinary differential equations.

the method is applicable to many bifurcation situations, of which Hopf, is the most important ones. They are known to occur in many physical systems and often play an important role for the systems longterm behavior.

We can note that the proposed method may be more efficient in term of approximation the nonlinear function is no regular or the equilibrium point is no regular. In this case, one cannot derive the nonlinear function and consequently one cannot study the linearized equation see [13-14]. In contrast to common analytical techniques based on eigenvalue computation (which can only be applied to systems of size dimension N ≤ 4), the method is applicable for systems of intermediate size because it is possible to compute numerically the optimal linear matrix and the roots of their characteristic equation (eigenvalues). the proposed linearization representing also a numerical confirmations of the prediction behaviour. Therefore it represent a good approximation to the initial nonlinear system.

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