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Near Rough and Near Exact Subgraphs in G_m-Closure Spaces

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Abstract

The basic concepts of some near open subgraphs, near rough, near exact and near fuzzy graphs are introduced and sufficiently illustrated. The G_m-closure space induced by closure operators is used to generalize the basic rough graph concepts. We introduce the near exactness and near roughness by applying the near concepts to make more accuracy for definability of graphs. We give a new definition for a membership function to find near interior, near boundary and near exterior vertices. Moreover, proved results, examples and counter examples are provided. The G_m-closure structure which suggested in this paper opens up the way for applying rich amount of topological facts and methods in the process of granular computing.

Key words*: Graph Theory, G_m-closure space, Near rough graphs, near exact graphs, Near fuzzy graphs, Near rough membership function.*

(2000) Math. Subject Classification: 54C05

1. Introduction

The notions of closure operator and closure system are very useful tools in several sections of mathematics. As an example, in algebra [5, 7], topology [8, 13, 14] and computer science theory [23, 28]. Many works have appeared recently for example in structural analysis [24, 25], in chemistry [26], and physics [11]. The theory of rough sets, proposed by Pawlak [20], is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. Using the concepts of lower and upper approximation in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. This leaded several authors to investigate about the closure systems and the closure operators in the framework of fuzzy set theory. As an example, see [4, 10, 23, 28]. The purpose of the present work is to put a starting point for the application of abstract topological graph theory in the rough set analysis. Also, we shall integrate some ideas in terms of concept in topological graph theory. Topological graph theory is a branch of Mathematics, whose concepts exists not only in almost all branches of Mathematics, but also in many real life application. We believe that topological graph structure will be an important base for modification of knowledge extraction and processing.

2. Preliminaries

This section presents a review of some fundamental notions of G_m-closure spaces [24, 25] and Pawlak's rough sets [6, 20, 21].

2.1 Fundamental Notions of G_m-Closure Spaces

In this section, we introduce the concepts of closure operators on digraphs, several known topological property on the obtained G_m-closure spaces are studies.

Definition 2.1.1. [24, 25] Let G = (V(G), E(G)) be a digraph, P(V(G)) its power set of all subgraphs of G and $Cl_G : P(V(G)) \to P(V(G))$ is a mapping associating with each subgraph H = (V(H), E(H)) a subgraph $Cl_G(V(H)) \subseteq V(G)$ called the closure subgraph of H such that:

$Cl_G(V(H)) = V(H) \cup \{v \in V(G) - V(H) \; ; \; \overrightarrow{hv} \in E(G)$ for all $h \in V(H)\}$.

The operation Cl_G is called graph closure operator and the pair $(G, \mathcal{F}_G)$ is called G-closure space, where $\mathcal{F}_G$ is the family of elements of Cl_G. Evidently $Cl_G(V(H)) = \cap\{V(F) \; ; \; V(F) \in \mathcal{F}_G$ and $V(H) \subseteq V(F)\}$. The dual of the graph closure operator Cl_G is the graph interior operator $Int_G : P(V(G)) \to P(V(G))$ defined by $Int_G(V(H)) = V(G) - Cl_G(V(G) - V(H))$ for all subgraph $H \subseteq G$. A family of elements of Int_G is called interior subgraph of H and denoted by τ_G. Clear that (G, τ_G) is a topological space. Evidently $Int_G(V(H)) = \cup \{V(O) \; ; \; V(F) \in \tau_G$ and $V(O) \subseteq V(H)\}$. Then the domain of Cl_G is equal to the domain of Int_G and also $Cl_G(V(H)) = V(G) - Int_G(V(G) - V(H))$. A subgraph H of G-closure space (G, τ_G) is called closed subgraph if

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$Cl_G(V(H)) = V(H)$. It is called open subgraph if its complement is closed subgraph, i.e., $Cl_G(V(G) - V(H)) = V(G) - V(H)$, or equivalently $Int_G(V(H)) = V(H)$.

Example 2.1.1. Let G = (V(G), E(G)) be a digraph such that: $V(G) = \{v_1, v_2, v_3, v_4\}$,
$E(G) = \{(v_1, v_2), (v_1, v_3), (v_2, v_1), (v_2, v_3), (v_4, v_3)\}$.

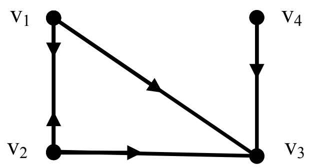

Fig. 1 Graph G given in Example 2.1.1.

Table 1: Cl_G for all subgraph $H \subseteq G$.

V(H)	Cl_G(V(H))	V(H)	Cl_G(V(H))
V(G)	V(G)	$\{v_1,v_4\}$	V(G)
ϕ	ϕ	$\{v_2,v_3\}$	$\{v_1,v_2,v_3\}$
$\{v_1\}$	$\{v_1,v_2,v_3\}$	$\{v_2,v_4\}$	V(G)
$\{v_2\}$	$\{v_1,v_2,v_3\}$	$\{v_3,v_4\}$	$\{v_3,v_4\}$
$\{v_3\}$	$\{v_3\}$	$\{v_1,v_2,v_3\}$	V(G)
$\{v_4\}$	$\{v_3,v_4\}$	$\{v_1,v_2,v_4\}$	V(G)
$\{v_1,v_2\}$	$\{v_1,v_2,v_3\}$	$\{v_1,v_3,v_4\}$	V(G)
$\{v_1,v_3\}$	$\{v_1,v_2,v_3\}$	$\{v_2,v_3,v_4\}$	V(G)

$\mathcal{F}_G = \{V(G), \phi, \{v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}\}$,

$\mathcal{T}_G = \{V(G), \phi, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}$.

We obtain a new definition to construct topological closure spaces from G-closure spaces by redefine graph closure operator on the resultant subgraphs as a domain of the graph closure operator and stop when the operator transfers each subgraph to itself.

Definition 2.1.2. [24, 25] Let G = (V(G), E(G)) be a digraph and $Cl_{Gm} : P(V(G)) \rightarrow P(V(G))$ an operator such that:

(a) It is called G_m-closure operator if $Cl_{Gm}(V(H)) = Cl_G(Cl_G(\ldots\ Cl_G(V(H))))$, m-times, for every subgraph $H \subseteq G$,

(b) it is called G_m-topological closure operator if $Cl_{Gm+1}(V(H)) = Cl_{Gm}(V(H))$ for all subgraph $H \subseteq G$.

The space $(G, \mathcal{F}_{Gm})$ is called G_m-closure space.

Example 2.1.2. Let G = (V(G), E(G)) be a digraph such that: $V(G) = \{v_1, v_2, v_3, v_4\}$,
$E(G) = \{(v_1, v_3), (v_2, v_1), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$.

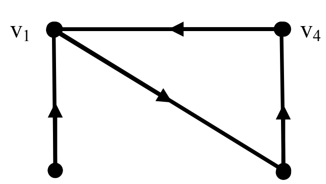

Fig. 2 Graph G given in Example 2.1.2.

Table 2: Cl_G and Cl_{G2} for all subgraph $H \subseteq G$.

V(H)	Cl_G(V(H))	Cl_{G2}(V(H))
V(G)	V(G)	V(G)
ϕ	ϕ	ϕ
$\{v_1\}$	$\{v_1, v_3\}$	$\{v_1, v_3, v_4\}$
$\{v_2\}$	$\{v_1, v_2, v_3\}$	V(G)
$\{v_3\}$	$\{v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_4\}$	$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_1, v_2\}$	$\{v_1, v_2, v_3\}$	V(G)
$\{v_1, v_3\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_2, v_3\}$	V(G)	V(G)
$\{v_2, v_4\}$	V(G)	V(G)
$\{v_3, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_1, v_2, v_3\}$	V(G)	V(G)
$\{v_1, v_2, v_4\}$	V(G)	V(G)
$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_2, v_3, v_4\}$	V(G)	V(G)
$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_2, v_3\}$	V(G)	V(G)

$\mathcal{F}_{G2} = \{V(G), \phi, \{v_1, v_3, v_4\}\}$,

$\mathcal{T}_{G2} = \{V(G), \phi, \{v_2\}\}$.

Proposition 2.1.1. [24] Let $(G, \mathcal{F}_{Gm})$ be a G_m-closure space. If H and K are two subgraphs of G such that $H \subseteq K \subseteq G$, then
$Cl_{Gm}(V(H)) \subseteq Cl_{Gm}(V(K))$ and $Int_{Gm}(V(H)) \subseteq Int_{Gm}(V(K))$.

Proposition 2.1.2. [24] Let $(G, \mathcal{F}_{Gm})$ be a G_m-closure space. If H and K are two subgraphs of G, then

(a) $Cl_{Gm}(V(H) \cup V(K)) = Cl_{Gm}(V(H)) \cup Cl_{Gm}(V(K))$.

(b) $Int_{Gm}(V(H) \cap V(K)) = Int_{Gm}(V(H)) \cap Int_{Gm}(V(K))$.

Proposition 2.1.3. [24] Let $(G, \mathcal{F}_{Gm})$ be a G_m-closure space. If H and K are two subgraphs of G, then

(a) $Cl_{Gm}(V(H) \cap V(K)) \subseteq Cl_{Gm}(V(H)) \cap Cl_{Gm}(V(K))$, and

(b) $Int_{Gm}(V(H)) \cup Int_{Gm}(V(K)) \subseteq Int_{Gm}(V(H) \cup V(K))$.

Remark 2.2.1. The converse of proposition (2.1.3) above need not be true in general, as the following example (2.3 in [24]).

Definition 2.1.3. [24] Let $(G, \mathcal{F}_{Gm})$ be a G_m-closure space and $H \subseteq G$, the boundary of H is denoted by $Bd_{Gm}(V(H))$ and is defined by

$$Bd_{Gm}(V(H)) = Cl_{Gm}(V(H)) - Int_{Gm}(V(H)).$$

Proposition 2.1.4. [24] Let (G, $\digamma_{Gm}$) be a G_m-closure space and $H \subseteq G$, then

(a) $Bd_{Gm}(V(H)) = Cl_{Gm}(V(H)) \cap Cl_{Gm}(V(G) - V(P))$.
(b) $Bd_{Gm}(V(P)) = Bd_{Gm}(V(G) - V(P))$.
(c) $Cl_{Gm}(V(P)) = V(P) \cup Bd_{Gm}(V(P))$.
(d) $Int_{Gm}(V(P)) = V(P) - Bd_{Gm}(V(P))$.

By a similar way of definitions of regular open set [27], semi-open set [16], pre-open set [18], γ-open set [9] (b-open set [2]), α-open set [15], and β-open set [1] (=semi-pre-open set [3]). We introduce the following definitions which are essential for our present study. In G_m-closure space (G, $\digamma_{Gm}$) the subgraph H in (G, $\digamma_{Gm}$) is called

(a) Regular open subgraph [24] (briefly R-osg) if $V(H) = Int_{Gm}(Cl_{Gm}(V(H)))$.
(b) Semi-open subgraph [24] (briefly S-osg) if $V(H) \subseteq Cl_{Gm}(Int_{Gm}(V(H)))$.
(c) Pre-open subgraph [24] (briefly P-osg) if $V(H) \subseteq Int_{Gm}(Cl_{Gm}(V(H)))$.
(d) γ-open subgraph (briefly γ-osg) if $V(H) \subseteq Cl_{Gm}(Int_{Gm}(V(H))) \cup Int_{Gm}(Cl_{Gm}(V(H)))$.
(e) α-open subgraph [24] (briefly α-osg) if $V(H) \subseteq Int_{Gm}(Cl_{Gm}(Int_{Gm}(V(H))))$.
(f) β-open subgraph [24] (briefly β-osg) if $V(H) \subseteq Cl_{Gm}(Int_{Gm}(Cl_{Gm}V(H)))$.

The complement of an R-osg (resp. S-osg, P-osg, γ-osg, α-osg and β-osg) is called R-closed subgraph (briefly R-csg) (resp. S-csg, P-csg, γ-csg, α-csg and β-csg).
The family of all R-osgs (resp. S-osgs, P-osgs, γ-osgs, α-osgs and β-osgs) of (G, $\digamma_{Gm}$) is denoted by $RO_{Gm}(G)$ (resp. $SO_{Gm}(G)$, $PO_{Gm}(G)$, $\gamma O_{Gm}(G)$, $\alpha O_{Gm}(G)$ and $\beta O_{Gm}(G)$). All of $SO_{Gm}(G)$, $PO_{Gm}(G)$, $\gamma O_{Gm}(G)$, $\alpha O_{Gm}(G)$ and $\beta O_{Gm}(G)$ are larger than τ_{Gm} and closed under forming arbitrary union.
The family of all R-csgs (resp. S-csgs, P-csgs, γ-csgs, α-csgs and β-csgs) of (G, $\digamma_{Gm}$) is denoted by $RC_{Gm}(G)$ (resp. $SC_{Gm}(G)$, $PC_{Gm}(G)$, $\gamma C_{Gm}(G)$, $\alpha C_{Gm}(G)$ and $\beta C_{Gm}(G)$).
The near closure (resp. near interior and near boundary)of a subgraph H of G in a G_m-closure space (G, $\digamma_{Gm}$) is denoted by $Cl^{j}_{Gm}(V(H))$ (resp. $Int^{j}_{Gm}(V(H))$ and $Bd^{j}_{Gm}(V(H))$) and defined by

$Cl^{j}_{Gm}(V(H)) = \cap\{V(F)\ ;\ V(F)$ is j-csg and $V(H) \subseteq V(F)\}$.
(resp. $Int^{j}_{Gm}(V(H)) = V(G) - Cl^{j}_{Gm}(V(G) - V(H))$ and $Bd^{j}_{Gm}(V(H)) = Cl^{j}_{Gm}(V(H)) - Int^{j}_{Gm}(V(H))$) where $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Proposition 2.1.5. [24] Let (G, $\digamma_{Gm}$) be G_m-closure space, the implication τ_{Gm} and the families of near open and near closed graphs are given by following statements.

(a) $RO_{Gm}(G) \subseteq \tau_{Gm} \subseteq \alpha O_{Gm}(G) \subseteq SO_{Gm}(G) \subseteq \gamma O_{Gm}(G) \subseteq \beta O_{Gm}(G)$,
(b) $\alpha O_{Gm}(G) \subseteq PO_{Gm}(G) \subseteq \gamma O_{Gm}(G)$,
(c) $RC_{Gm}(G) \subseteq \digamma_{Gm} \subseteq \alpha C_{Gm}(G) \subseteq SC_{Gm}(G) \subseteq \gamma C_{Gm}(G) \subseteq \beta C_{Gm}(G)$,
(d) $\alpha C_{Gm}(G) \subseteq PC_{Gm}(G) \subseteq \gamma C_{Gm}(G)$.

2.2. Fundamental Notions of Uncertainty

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space K = (X, R), where X is a set called the universe and R is an equivalence relation [17, 21]. The equivalence classes of R are also known as the granules, elementary sets or blocks, we shall use $R_x \subseteq X$ to denote the equivalence class containing $x \in X$. In the approximation space, we consider two operators, the upper and lower approximations of subsets: Let $A \subseteq X$, then the lower approximation (resp. the upper approximation) of A is given by

$$L(A) = \{x \in X : R_x \subseteq A\}$$
$$(\text{resp. } U(A) = \{x \in X : R_x \cap A \neq \phi\})$$

Boundary, positive and negative regions are also defined:

$$Bd_R(A) = U(A) - L(A),$$
$$POS_R(A) = L(A),$$
$$NEG_R(A) = X - U(A).$$

These notions can be also expressed by rough membership functions [21], namely,

$$\mu^R_A(x) = \frac{|R_x \cap A|}{|R_x|}, \quad x \in X.$$

Different values defines boundary ($0 < \mu^R_A(x) < 1$), positive ($\mu^R_A(x) = 1$) and negative ($\mu^R_A(x) = 0$) regions. The membership function is a kind of conditional probability and its value can be interpreted as a degree of uncertainty to which x belongs to A.

Fuzzy set [29] is a way to represent populations that the set theory cannot describe definitely, fuzzy sets use a many (usually infinite) valued membership function, unlike classical set theory which uses a two valued membership function (i.e. an element is either in a set or it is not). Let X denotes a universal set and $A \subseteq X$. Then a membership function on X, μ_A, is a function;

$\mu_A : X \to L$ for some partially order set L.

L usually is a lattice [11]. Intuitively the membership function, μ_A, gives the degree to which an element $x \in X$ is in the fuzzy set A. In the case L is the closed interval [0, 1], we call it the Standard Fuzzy Set Theory.

2. Near Rough and Near Exact Subgraphs in G_m-closure Spaces

The present section is devoted to introduce the near exactness and near roughness by applying the concepts of near open subgraphs to make more accuracy for definability of graphs. Let H be a subgraph of a graph G. Let $Int_{Gm}(V(H))$, $Cl_{Gm}(V(H))$ and $Bd_{Gm}(V(H))$ be G_m-closure, G_m-interior and G_m-boundary region respectively. H is G_m-exact if $Bd_{Gm}(V(H)) = \phi$ otherwise H is G_m-rough [14]. We shall express near G_m-rough graph properties in terms of G_m-topological closure concepts. Let $Cl^{j}_{Gm}(V(H))$, $Int^{j}_{Gm}(V(H))$ and $Bd^{j}_{Gm}(V(H))$ be near G_m-closure, near G_m-interior, and near G_m-boundary vertices respectively, where $j\in\{R, S, P, \gamma, \alpha, \beta\}$. H is a near G_m-exact (briefly ${}_jG_m$-exact) graph if $Bd^{j}_{Gm}(V(H)) = \phi$, otherwise H is a near G_m-rough (briefly ${}_jG_m$-rough). It is clear H is ${}_jG_m$-exact iff $Cl^{j}_{Gm}(V(H)) = Int^{j}_{Gm}(V(H))$. In Pawlak space a subset $A \subseteq X$ has two possibilities rough or exact. The following definition introduces new types of near definability for a subgraph $H \subseteq G$ in a G_m-closure space (G, F_{Gm}).

Definition 3.1. Let (G, F_{Gm}) be a G_m-closure space and $H \subseteq G$, then H is called

(a) totally ${}_jG_m$-definable (${}_jG_m$-exact) graph if $Int^{j}_{Gm}(V(H)) = V(H) = Cl^{j}_{Gm}(V(H))$,

(b) internally ${}_jG_m$-definable graph if $Int^{j}_{Gm}(V(H)) = V(H)$, $Cl^{j}_{Gm}(V(H)) \neq V(H)$,

(c) externally ${}_jG_m$-definable graph if $Int^{j}_{Gm}(V(H)) \neq V(H)$, $Cl^{j}_{Gm}(V(H)) = V(H)$,

(d) ${}_jG_m$-indefinable (${}_jG_m$-rough) graph if $Int^{j}_{Gm}(V(H)) \neq V(H)$, $Cl^{j}_{Gm}(V(H)) \neq V(H)$, where $j\in\{R, S, P, \gamma, \alpha, \beta\}$.

Proposition 3.1. Let (G, F_{Gm}) be a G_m-closure space and H be a subgraph of G. If H is G_m-exact graph, then it is ${}_jG_m$-exact for all $j\in\{R, S, P, \gamma, \alpha, \beta\}$

Proof. The proofs of the six cases are similar; So, we will only prove the case when $j = \gamma$: Let H be G_m-exact graph, then $Cl_{Gm}(V(H)) = V(H) = Int_{Gm}(V(H))$. Now,

$$\begin{aligned} Cl_{Gm}(V(H)) &= \cap \{V(F) ; V(F) \in F_{Gm} \text{ and } V(H) \subseteq V(F)\} \\ &\supseteq \cap\{V(F) ; V(F) \in \gamma C_{Gm}(G) \text{ and } V(H) \subseteq V(F)\} \\ &\quad \text{since } F_{Gm} \subseteq \gamma C_{Gm}(G) \\ &= Cl^{\gamma}_{Gm}(V(H)) \qquad (3.1.1) \end{aligned}$$

$$\begin{aligned} \text{Also, } Int_{Gm}(V(H)) &= V(G) - Cl_{Gm}(V(G) - V(H)) \\ &\subseteq V(G) - Cl^{\gamma}_{Gm}(V(G) - V(H)) \\ &\quad \text{since } T_{Gm} \subseteq \gamma O_{Gm}(G) \\ &= Int^{\gamma}_{Gm}(V(H)) \qquad (3.1.2) \end{aligned}$$

From (3.1.1) and (3.1.2) we get $Int_{Gm}(V(H)) \subseteq Int^{\gamma}_{Gm}(V(H)) \subseteq V(H) \subseteq Cl^{\gamma}_{Gm}(V(H)) \subseteq Cl_{Gm}(V(H))$. Since H is exist we get $Int^{\gamma}_{Gm}(V(H)) = V(H) = Cl^{\gamma}_{Gm}(V(H))$. Hence H is ${}_{\gamma}G_m$-exact.

The converse of the above proposition is not true in general as the following example illustrates.

Example 3.1. Let G = (V(G), E(G)) be a digraph such that: $V(G) = \{v_1, v_2, v_3, v_4\}$,
$E(G) = \{(v_2, v_3), (v_3, v_4), (v_4, v_2)\}$.

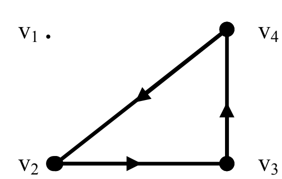

Fig. 3 Graph G given in Example 3.1.

$F_{G2} = \{V(G), \phi, \{v_1\}, \{v_2, v_3, v_4\}\}$,

$T_{G2} = \{V(G), \phi, \{v_1\}, \{v_2, v_3, v_4\}\}$.

Let H = (V(H), E(H)); $V(H) = \{v_1, v_2\}$, $E(H) = \phi$. Then $Int_{G2}(V(H)) = \{v_1\}$ and $Cl_{G2}(V(H)) = V(G)$, that is, H is a G_2-rough graph. But

$Int^{\beta}_{G2}(V(H)) = V(H) = Cl^{\beta}_{G2}(V(H))$, that is, H is ${}_{\beta}G_2$-exact graph.

In a G_m-topological closure space (G, τ_{Gm}), we shall use $\mathrm{Int}^{j}_{Gm}(V(H))|_{\tau Gm}$ (resp. $\mathrm{Cl}^{j}_{Gm}(V(H))|_{\tau Gm}$ and $\mathrm{Bd}^{j}_{Gm}(V(H))|_{\tau Gm}$) for a subgraph $H \subseteq G$ to denote $\mathrm{Int}^{j}_{Gm}(V(H))$ (resp. $\mathrm{Cl}^{j}_{Gm}(V(H))$ and $\mathrm{Bd}^{j}_{Gm}(V(H))$) with respect to the G_m-topology τ_{Gm} for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Proposition 3.2. Let $(G, \mathcal{F}_{Gm})$ and $(G', \mathcal{F}'_{Gm})$ be two G_m-closure space such that the family of ${}_jG_m$-open subgraphs in τ_{Gm} subset of the family of ${}_jG_m$-open subgraphs in τ'_{Gm} for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$. If $H \subseteq G$, G' is ${}_jG_m$-exact in $(G, \mathcal{F}_{Gm})$ then H is ${}_jG_m$-exact in $(G', \mathcal{F}'_{Gm})$.

Proof. Since $\mathrm{Bd}^{j}_{Gm}(V(H))|_{\tau' Gm} \subseteq \mathrm{Bd}^{j}_{Gm}(V(H))|_{\tau Gm}$ and $\mathrm{Bd}^{j}_{Gm}(V(H))|_{\tau Gm} = \phi$ for all $j \in \{R,S,P,\gamma,\alpha,\beta\}$. Then $\mathrm{Bd}^{j}_{Gm}(V(H))|_{\tau' Gm} = \phi$ and H is ${}_jG_m$-exact with respect to τ'_{Gm}.

In Proposition (3.2), it is not necessary for τ_{Gm} to be coarser than τ'_{Gm}, also the converse of this proposition is not true in general as the following example illustrates.

Example 3.2. Let $G = (V(G), E(G))$ and $G' = (V(G), E(G'))$ be two digraph such that $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_1), (v_2, v_3), (v_4, v_3)\}$ and $E(G') = \{(v_2, v_3), (v_3, v_4), (v_4, v_2)\}$.

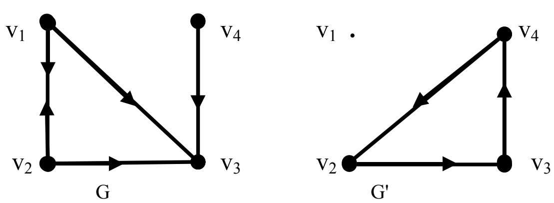

Fig. 4 Graph G and G' given in Example 3.2.

$\mathcal{F}_{G1} = \{V(G), \phi, \{v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}\}$,

$\tau_{G1} = \{V(G), \phi, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}$.

$\mathcal{F}'_{G2} = \{V(G), \phi, \{v_1\}, \{v_2, v_3, v_4\}\}$,

$\tau'_{G2} = \{V(G), \phi, \{v_1\}, \{v_2, v_3, v_4\}\}$.

Then $PO_{G1}(G)|_{\tau G1} \subseteq PO_{G2}(G')|_{\tau' G2}$.
If $H = (V(H), E(H))$; $V(H)=\{v_4\}$, $E(H) = \phi$, $\mathrm{Bd}^{P}_{G1}(V(H))|_{\tau G1} = \{v_3\}$ and $\mathrm{Bd}^{P}_{G'2}(V(H))|_{\tau' G'2} = \phi$. Thus H is is ${}_PG_2$-exact graph in $\tau'_{G'2}$ but it is not ${}_PG_1$-exact in τ_{G1}.

Lemma 3.1. Let $(G, \mathcal{F}_{Gm})$ and $(G', \mathcal{F}'_{Gm})$ be two G_m-closure space and H is a subgraph of G, G'. Then $\mathrm{Cl}^{j}_{Gm}(V(H))|_{\tau' Gm} = \mathrm{Cl}^{j}_{Gm}(V(H))|_{\tau Gm}$ if and only if $\mathrm{Int}^{j}_{Gm}(V(H))|_{\tau' Gm} = \mathrm{Int}^{j}_{Gm}(V(H))|_{\tau Gm}$ for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Proof. The proofs of the six cases are similar; So, we will only prove the case when $j = \alpha$: Now,

$\mathrm{Cl}^{\alpha}_{Gm}(V(H))|_{\tau' Gm} = \mathrm{Cl}^{\alpha}_{Gm}(V(H))|_{\tau Gm}$ if and only if

$\cap\ \{V(F) \in V(G); V(H) \subseteq V(F), V(F) \in \alpha C_{Gm}(G')$ with respect to $\tau'_{G'}\}$

$= \cap\ \{V(F) \in V(G); V(H) \subseteq V(F), V(F) \in \alpha C_{Gm}(G)$ with respect to $\tau_G\}$

if and only if

$V(G) - \cap\ \{V(F) \in V(G); V(H) \subseteq V(F), V(F) \in \alpha C_{Gm}(G')$ with respect to $\tau'_{G'}\}$

$= V(G) - \cap\ \{V(F) \in V(G); V(H) \subseteq V(F), V(F) \in \alpha C_{Gm}(G)$ with respect to $\tau_G\}$

if and only if

$\cup\{V(G)-V(F) \subseteq V(G); V(G)-V(H) \supseteq V(G)-V(F),\ V(G)-V(F) \in \alpha O_{Gm}(G')$ w. r. t. $\tau'_{G'}\}$

$= \cup\{V(G)-V(F) \subseteq V(G); V(G)-V(H) \supseteq V(G)-V(F),$ $V(G)-V(F) \in \alpha O_{Gm}(G)$ w. r. t. $\tau_G\}$

if and only if

$\mathrm{Int}^{j}_{Gm}(V(H))|_{\tau' Gm} = \mathrm{Int}^{j}_{Gm}(V(H))|_{\tau Gm}$.

Let us observe Lemma 3.1. The following proposition gives the condition for ${}_jG_m$-exact graphs in $(G', \mathcal{F}'_{Gm})$ to be ${}_jG_m$-exact graphs in $(G, \mathcal{F}_{Gm})$, where the family of ${}_jG_m$-open graphs in $(G, \mathcal{F}_{Gm})$ subset of the family of ${}_jG_m$-open graphs in $(G', \mathcal{F}'_{Gm})$ for all $j \in \{R,S,P,\gamma, \alpha, \beta\}$.

Proposition 3.3. Let $(G, \mathcal{F}_{Gm})$ and $(G', \mathcal{F}'_{Gm})$ be two G_m-closure space such that the family of ${}_jG_m$-open subgraphs in τ_{Gm} subset of the family of ${}_jG_m$-open subgraphs in τ'_{Gm} for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$. Then each ${}_jG_m$-exact graph in $(G', \mathcal{F}'_{G'm})$ is ${}_jG_m$-exact in $(G, \mathcal{F}$

$_{Gm}$) if and only if $Cl^{j}_{Gm}(V(H))|_{\tau Gm} = Cl^{j}_{Gm}(V(H))|_{\tau' G'm}$ for all subgraph $H \subseteq G, G'$.

Proof. If H is $_jG_m$-exact graph in (G', $\digamma'_{G'm}$) for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$, then $Cl^{j}_{Gm}(V(H))|_{\tau' G'm} = V(H)$ and $Cl^{j}_{Gm}(V(H))|_{\tau Gm} = V(H)$, hence $Cl^{j}_{Gm}(V(H))|_{\tau Gm} = Cl^{j}_{Gm}(V(H))|_{\tau' G'm}$. Conversely, if $Cl^{j}_{Gm}(V(H))|_{\tau Gm} = Cl^{j}_{Gm}(V(H))|_{\tau' G'm}$ and H is $_jG_m$-exact in (G', $\digamma'_{G'm}$). Then H is $_jG_m$-exact in (G, $\digamma_{Gm}$).

3. Near Rough Membership Function in G_m-closure Spaces

Original rough membership function is defined using equivalence classes. It was extended to topological spaces [15], namely. If τ_{Gm} is a G_m-topology on a finite graph G, then the G_m-rough membership function for subgraph H on G is

$$\mu^{H}_{Gm}(v) = \frac{|\{\bigcap V(K_v)\} \bigcap V(H))|}{|\{\bigcap V(K_v)\}|}, \; v \in G \quad (4.1)$$

where K_v is any suggraph of τ_{Gm} containing v.

In equation (4.1), since it is not necessary for $\{ \cap K_v \}$ to be a j-open graph for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$, then we cannot use this equation to express j-boundary region, j-interior and j-exterior of a subgraph $H \subseteq G$ in a G_m-closure space (G, $\digamma_{Gm}$) even thought K_v is a j-open graph.

Example 4.1. Let (G, $\digamma_{Gm}$) be a G_m-closure space which is given in Example (2.1.1).

$\digamma_{G1} = \{V(G), \phi, \{v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}\}$,

$\tau_{G1} = \{V(G), \phi, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}$,

$\beta O_{G1}(G) = \{V(G), \phi \{v_1\}, \{v_2\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\} \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}$.

If H = (V(H), E(H)); $V(H)=\{v_1, v_3, v_4\}$, $E(H)= \{(v_1, v_3), (v_4, v_3)\}$, in equation (4.1), then

$$\mu^{H}_{Gm}(v_1) = \frac{|\{\bigcap V(K_{v_1})\} \bigcap \{v_1, v_3, v_4\}|}{|\{\bigcap V(K_{v_1})\}|} = \frac{|\{v_1, v_2\} \bigcap \{v_1, v_3, v_4\}|}{|\{v_1, v_2\}|} = \frac{1}{2},$$

That is $v_1 \notin Int_{Gm}(V(H))$; but $Int^{\beta}_{Gm}(V(H)) = \{v_1, v_3, v_4\}$.

Also, in the case of $K_v \subseteq \beta O_{Gm}(G)$ and H = (V(H), E(H)); $V(H) = \{v_3\}$, $E(H) = \phi$, then

$$\mu^{H}_{Gm}(v_3) = \frac{|\{\bigcap V(K_{v_3})\} \bigcap \{v_3\}|}{|\{\bigcap V(K_{v_3})\}|} = \frac{|\{v_3\} \bigcap \{v_3\}|}{|\{v_3\}|} = 1; \text{ but}$$

$Int^{\beta}_{Gm}(V(H)) = \phi$.

In a G_m-closure space (G, $\digamma_{Gm}$), we use $_jG_m$-boundary region "briefly $_jBd_{Gm}(V(H))$" (resp. $_jG_m$-positive region "briefly $_jPOS_{Gm}(V(H))$" and $_jG_m$-negative region "briefly $_jNEG_{Gm}(V(H)))$ to denote $Bd^{j}_{Gm}(V(H))$ (resp. $Int^{j}_{Gm}(V(H))$ and $Ext^{j}_{Gm}(V(H))$) for a subgraph $H \subseteq G$, where $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

we introduce the following definition for a $_jG_m$-rough membership function to express $_jBd_{Gm}(V(H))$, $_jPOS_{Gm}(V(H)))$ and $_jNEG_{Gm}(V(H)))$ for a subgraph $H \subseteq G$, where $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Definition 4.1. Let (G, $\digamma_{Gm}$) be a G_m-closure space and $H \subseteq G$. Then the near G_m-rough (briefly $_jG_m$-rough) membership function on G is $_j\mu^{H}_{Gm} : G \to [0, 1]$ and it is given by

$$_j\mu^{H}_{Gm}(v) = \begin{Bmatrix} 1 & \text{if } 1 \in {_jK_v}(V(H)), \\ \min {_jK_v}(V(H)) & \text{otherwise} \end{Bmatrix}$$

where $_jK_v(V(H)) =$

$$\left\{\frac{|V(K) \bigcap V(H)|}{|V(K)|} : K \text{ is a } {_jGm}-\text{open graph}, v \in V(K)\right\}$$

for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Theorem 4.1. Let (G, $\digamma_{Gm}$) be a G_m-closure space and $H \subseteq G$. Then

(a) $v \in Int^{j}_{Gm}(V(H))$ if and only if $_j\mu^{H}_{Gm}(v) = 1$,

(b) $v \in Bd^{j}_{Gm}(V(H))$ if and only if $0 < {_j\mu^{H}_{Gm}}(v) < 1$,

(c) $v \in$ is a $_jG_m$-exterior vertex of H (briefly $v \in EXT^{j}_{Gm}(V(H))$) if and only if $_j\mu^{H}_{Gm}(v) = 0$.

For all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Proof. The proofs of the six cases are similar; So, we will only prove the case when $j = \beta$:

(a) $v \in Int^{\beta}_{Gm}(V(H))$ iff $\exists$ $K \in \beta O_{Gm}(G)$ such that $v \in V(K) \subseteq V(H)$

iff $\exists$ $K \in \beta O_{Gm}(G)$, $v \in V(K)$ such that $\frac{|V(K) \bigcap V(H)|}{|V(K)|} = 1$

iff $_\beta\mu^{H}_{Gm}(v) = 1$.

(b) $v \in Bd^{\beta}_{Gm}(V(H))$ iff $\forall$ $K \in \beta O_{Gm}(G)$, $v \in V(K)$, we have $V(K) \cap V(H) \neq \phi$ and $V(K) \cap (V(G) - V(H)) \neq \phi$
iff $\forall$ $K \in \beta O_{Gm}(G)$, $v \in V(K)$, we have $0 < |V(K) \cap V(H)| < |V(K)|$
iff $\forall$ $K \in \beta O_{Gm}(G)$, $v \in V(K)$, we have $0 < \frac{|V(K) \cap V(H)|}{|V(K)|} < 1$
iff $0 < {}_{j}\mu^{H}_{Gm}(v) < 1$.

(c) $v \in EXT^{\beta}_{Gm}(V(H))$ iff $\exists$ $K \in \beta O_{Gm}(G)$ such that $v \in V(K) \subseteq (V(G) - V(H))$
iff $\exists$ $K \in \beta O_{Gm}(G)$, $v \in V(K)$ such that $V(K) \cap V(H) = \phi$
iff $\exists$ $K \in \beta O_{Gm}(G)$, $v \in V(K)$ such that $\frac{|V(K) \cap V(H)|}{|V(K)|} = 0$
iff ${}_{\beta}\mu^{H}_{Gm}(v) = 0$.

The ${}_{j}G_m$-rough membership function defines ${}_{j}Bd_{Gm}(V(H))$ (resp. ${}_{j}POS_{Gm}(V(H))$ and ${}_{j}NEG_{Gm}(V(H))$ if $0 < {}_{j}\mu^{H}_{Gm}(v) < 1$ (resp. ${}_{j}\mu^{H}_{Gm}(v) = 1$ and ${}_{j}\mu^{H}_{Gm}(v) = 0$) for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ in a G_m-closure space (G, F_{Gm}) and $H \subseteq G$. The following example illustrates Theorem (4.1) for $j \in \{\gamma, \alpha, \beta\}$.

Example 4.2. Let (G, F_{Gm}) be a G_m-closure space which is given in Example (2.1.1).

$F_{G1} = \{V(G), \phi, \{v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}\}$,

$T_{G1} = \{V(G), \phi, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}$,

$\gamma O_{G1}(G) = \{V(G), \phi, \{v_1\}, \{v_2\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}$,

$\alpha O_{G1}(G) = \{V(G), \phi, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}$, and

$\beta O_{G1}(G) = \{V(G), \phi, \{v_1\}, \{v_2\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\} \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}$.

If $H = (V(H), E(H))$; $V(H)=\{v_1, v_3\}$, $E(H)= \{(v_1, v_3)\}$, we get:

${}_{\gamma}\mu^{H}_{Gm}(v_1) = 1/3$, ${}_{\gamma}\mu^{H}_{Gm}(v_2) = 1/3$, ${}_{\gamma}\mu^{H}_{Gm}(v_3) = 1/2$, ${}_{\gamma}\mu^{H}_{Gm}(v_4) = 0$,

${}_{\alpha}\mu^{H}_{Gm}(v_1) = 1$, ${}_{\alpha}\mu^{H}_{Gm}(v_2) = 0$, ${}_{\alpha}\mu^{H}_{Gm}(v_3) = 1/3$, ${}_{\alpha}\mu^{H}_{Gm}(v_4) = 0$,

${}_{\beta}\mu^{H}_{Gm}(v_1) = 1$, ${}_{\beta}\mu^{H}_{Gm}(v_2) = 0$, ${}_{\beta}\mu^{H}_{Gm}(v_3) = 1$, ${}_{\beta}\mu^{H}_{Gm}(v_4) = 0$,

Therefore

$Int^{\gamma}_{Gm}(V(H)) = \{v_1\}$, $Bd^{\gamma}_{Gm}(V(H)) = \{v_3\}$, $EXT^{\gamma}_{Gm}(V(H)) = \{v_2, v_4\}$,

$Int^{\alpha}_{Gm}(V(H)) = \phi$, $Bd^{\alpha}_{Gm}(V(H)) = \{v_1, v_2, v_3\}$, $EXT^{\alpha}_{Gm}(V(H)) = \{v_4\}$,

$Int^{\beta}_{Gm}(V(H)) = \{v_1, v_3\}$, $Bd^{\beta}_{Gm}(V(H)) = \phi$, $EXT^{\beta}_{Gm}(V(H)) = \{v_2, v_4\}$.

4. Near Fuzzy Graphs in G_m-closure spaces

Near membership functions allow us to express fuzzy theory in G_m-closure spaces. In the following definition we define a near G_m-fuzzy (briefly ${}_{j}G_m$-fuzzy) graph by using the ${}_{j}G_m$-rough membership function of G_m-closure spaces for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Definition 5.1. Let (G, F_{Gm}) be a G_m-closure space and $H \subseteq G$. The ${}_{j}G_m$-fuzzy graph of H is denoted by ${}_{j}H^{f}$ and is given by

${}_{j}H^{f} = \{(v, {}_{j}\mu^{H}_{Gm}(v)) : \text{for all } v \in G\}$, $j \in \{R,S,P,\gamma,\alpha,\beta\}$.

Example 5.1. Let (G, F_{Gm}) be a G_m-closure space which is given in Example (2.1.1).
If $H = (V(H), E(H))$; $V(H)=\{v_1, v_3\}$, $E(H)= \{(v_1, v_3)\}$, then

${}_{\gamma}H^{f} = \{(v_1, 1), (v_2, 0), (v_3, 1/3), (v_4, 0)\}$,

${}_{\alpha}H^{f} = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$, and

${}_{\beta}H^{f} = \{(v_1, 1), (v_2, 0), (v_3, 1), (v_4, 0)\}$.

Now, we introduce some simple operations on ${}_{j}G_m$-fuzzy graphs for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Definition 5.2. Let H and K be two subgraphs of G in a G_m-closure space (G, F_{Gm}). We say that ${}_{j}H^{f}$ is included in ${}_{j}K^{f}$ (briefly ${}_{j}H^{f} \subseteq {}_{j}K^{f}$) for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ if and only if

$${}_{j}\mu^{H}_{Gm}(v) \leq {}_{j}\mu^{K}_{Gm}(v) \text{ for all } v \in G.$$

Definition 5.3. Let H and K be two subgraphs of G in a G_m-closure space (G, F_{Gm}). We say that ${}_{j}H^{f}$ and ${}_{j}K^{f}$ are equal (briefly ${}_{j}H^{f} = {}_{j}K^{f}$) for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ if and only if

$${}_{j}\mu^{H}_{Gm}(v) = {}_{j}\mu^{K}_{Gm}(v) \text{ for all } v \in G.$$

If at least one v of G is such that the equality ${}_{j}\mu^{H}_{Gm}(v) = {}_{j}\mu^{K}_{Gm}(v)$ is not satisfied, we say that ${}_{j}H^{f}$ and ${}_{j}K^{f}$ are not equal (briefly ${}_{j}H^{f} \neq {}_{j}K^{f}$) for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Definition 5.4. Let H and K be two subgraphs of G in a G_m-closure space (G, $\mathcal{F}_{Gm}$). We say that ${}_jH^f$ and ${}_jK^f$ are complementary (briefly ${}_jK^f = {}_jH^{fc}$) for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ if and only if

$${}_j\mu_{Gm}^{K}(v) = 1 - {}_j\mu_{Gm}^{H}(v) \text{ for all } v \in G.$$

One obviously has $({}_jH^{fc})^c = {}_jH^f$ for all $j \in \{R,S,P,\gamma,\alpha,\beta\}$.

Example 5.2. Let (G, $\mathcal{F}_{Gm}$) be a G_m-closure space which is given in Example (2.1.1).
If H = (V(H), E(H)); V(H)=$\{v_1, v_3\}$, E(H)= $\{(v_1, v_3)\}$, then
${}_\gamma H^f = \{(v_1, 1), (v_2, 0), (v_3, 1/3), (v_4, 0)\}$,
${}_\gamma H^{fc} = \{(v_1, 0), (v_2, 1), (v_3, 2/3), (v_4, 1)\}$.

Definition 5.5. Let H, K and M be subgraphs of G in a G_m-closure space (G, $\mathcal{F}_{Gm}$). We define the intersection ${}_jH^f \cap {}_jK^f$ for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ as the largest ${}_jG_m$-fuzzy graph contained at the same time in ${}_jH^f$ and ${}_jK^f$. that is, if ${}_jM^f = {}_jH^f \cap {}_jK^f$, then

$${}_j\mu_{Gm}^{M}(v) = \min\{{}_j\mu_{Gm}^{H}(v), {}_j\mu_{Gm}^{K}(v)\} \text{ for all } v \in G.$$

Definition 5.6. Let H, K and M be subgraphs of G in a G_m-closure space (G, $\mathcal{F}_{Gm}$). We define the union ${}_jH^f \cup {}_jK^f$ for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ as the smallest ${}_jG_m$-fuzzy graph contains both ${}_jH^f$ and ${}_jK^f$. that is, if ${}_jM^f = {}_jH^f \cup {}_jK^f$, then

$${}_j\mu_{Gm}^{M}(v) = \max\{{}_j\mu_{Gm}^{H}(v), {}_j\mu_{Gm}^{K}(v)\} \text{ for all } v \in G.$$

Example 5.3. Let (G, $\mathcal{F}_{Gm}$) be a G_m-closure space which is given in Example (2.1.1).
If H = (V(H), E(H)); V(H) =$\{v_1, v_3\}$, E(H) = $\{(v_1, v_3)\}$, and If K = (V(K), E(K)); V(K) = $\{v_1, v_2, v_3\}$, E(H) = $\{(v_1, v_2), (v_1, v_3), (v_2, v_1), (v_2, v_3)\}$, then

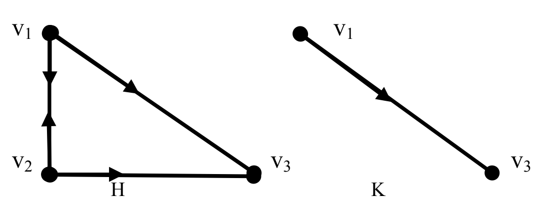

Fig. 5 Subgraph H and K given in Example 5.3.

${}_\alpha H^f = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$,
${}_\alpha K^f = \{(v_1, 1), (v_2, 1), (v_3, 3/4), (v_4, 0)\}$,
${}_\alpha H^f \cap {}_\alpha K^f = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$,
${}_\alpha H^f \cup {}_\alpha K^f = \{(v_1, 1), (v_2, 1), (v_3, 3/4), (v_4, 0)\}$.

Definition 5.7. Let H and K be two subgraphs of G in a G_m-closure space (G, $\mathcal{F}_{Gm}$). The disjunction sum of two ${}_jG_m$-fuzzy graph for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ is define in terms of union and intersections in the following fashion

$${}_jH^f \oplus {}_jK^f = ({}_jH^f \cap {}_jK^{fc}) \cup ({}_jH^{fc} \cap {}_jK^f).$$

Example 5.3. In Example (4.3)., we get
${}_\alpha H^f = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$,
${}_\alpha K^f = \{(v_1, 1), (v_2, 1), (v_3, 3/4), (v_4, 0)\}$. Hence
${}_\alpha H^{fc} = \{(v_1, 2/3), (v_2, 2/3), (v_3, 1/2), (v_4, 1)\}$,
${}_\alpha K^{fc} = \{(v_1, 0), (v_2, 0), (v_3, 1/4), (v_4, 1)\}$,
${}_\alpha H^f \cap {}_\alpha K^{fc} = \{(v_1, 0), (v_2, 0), (v_3, 1/4), (v_4, 0)\}$,
${}_\alpha H^{fc} \cap {}_\alpha K^f = \{(v_1, 2/3), (v_2, 2/3), (v_3, 1/2), (v_4, 0)\}$.
Thus
${}_\alpha H^f \oplus {}_\alpha K^f = ({}_\alpha H^f \cap {}_\alpha K^{fc}) \cup ({}_\alpha H^{fc} \cap {}_\alpha K^f) = \{(v_1, 2/3), (v_2, 2/3), (v_3, 1/2), (v_4, 0)\}$.

Definition 5.8. Let H and K be two subgraphs of G in a G_m-closure space (G, $\mathcal{F}_{Gm}$). The difference ${}_jH^f - {}_jK^f$ for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ is define by

$${}_jH^f - {}_jK^f = {}_jH^f \cap {}_jK^{fc}.$$

Of course, except in particular cases, ${}_jH^f - {}_jK^f = {}_jK^f - {}_jH^f$ for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

Example 5.5. In Example (4.3)., we get
${}_\alpha H^f = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$,
${}_\alpha K^f = \{(v_1, 1), (v_2, 1), (v_3, 3/4), (v_4, 0)\}$. Then
${}_\alpha H^f - {}_\alpha K^f = {}_\alpha H^f \cap {}_\alpha K^{fc} = \{(v_1, 0), (v_2, 0), (v_3, 1/4), (v_4, 0)\}$, since
${}_\alpha K^{fc} = \{(v_1, 0), (v_2, 0), (v_3, 1/4), (v_4, 1)\}$.

Definition 5.9. Let H and K be two subgraphs of G in a finite G_m-closure space (G, $\mathcal{F}_{Gm}$). The j-Hamming distance between H and K for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ is define by

$${}_jd(H, K) = \sum_{i=1}^{n} |{}_j\mu_{Gm}^{H}(v_i) - {}_j\mu_{Gm}^{K}(v_i)|.$$

Example 5.6. In Example (4.3)., we get
${}_\alpha H^f = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$, and
${}_\alpha K^f = \{(v_1, 1), (v_2, 1), (v_3, 3/4), (v_4, 0)\}$. Then

$${}_\alpha d(H, K) = |1/3 - 1| + |1/3 - 1| + |1/2 - 3/4| + |0 - 0|$$
$$= 2/3 + 2/3 + 1/4 + 0 = 1.583 .$$

Definition 5.10. Let H and K be two subgraphs of G in a finite G_m-closure space (G, $\mathcal{F}_{Gm}$). The j-Euclidean distance or j-quadratic distance between H and K for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ is define by

$$_{j}e(H, K) = \sqrt{\sum_{i=1}^{n}({}_{j}\mu_{Gm}^{H}(v_i) - {}_{j}\mu_{Gm}^{K}(v_i))^2} .$$

Definition 5.11. Let H and K be two subgraphs of G in a finite G_m-closure space (G, F_{Gm}). The j-Euclidean norm between H and K for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ is define by

$$_{j}e^2(H, K) = \sum_{i=1}^{n}({}_{j}\mu_{Gm}^{H}(v_i) - {}_{j}\mu_{Gm}^{K}(v_i))^2 .$$

Example 5.7. In Example (4.3)., we get
$_{\alpha}H^f = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$, and
$_{\alpha}K^f = \{(v_1, 1), (v_2, 1), (v_3, 3/4), (v_4, 0)\}$. Then
$_{j}e(H, K) = \sqrt{4/9+4/9+1/16+0} = \sqrt{0.951} = 0.975$, and
$_{j}e^2(H, K) = 4/9 + 4/9 + 1/16 + 0 = 0.951$.

Definition 5.12. Let H and K be two subgraphs of G in a finite G_m-closure space (G, F_{Gm}). The generalized relative j-Euclidean between H and K for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ is define by

$$_{j}\delta(H, K) = \frac{_{j}d(H, K)}{n} = \frac{1}{n}\sum_{i=1}^{n}|\,{}_{j}\mu_{Gm}^{H}(v_i) - {}_{j}\mu_{Gm}^{K}(v_i)| .$$

Example 5.8. In Example (4.3)., we get
$_{\alpha}H^f = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$, and
$_{\alpha}K^f = \{(v_1, 1), (v_2, 1), (v_3, 3/4), (v_4, 0)\}$. Then
$_{\alpha}\delta(H, K) = \frac{1.583}{4} = 0.396$.

Definition 5.13. Let H and K be two subgraphs of G in a finite G_m-closure space (G, F_{Gm}). The relative j-Euclidean between H and K for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ is define by

$$_{j}\varepsilon(H, K) = \frac{_{j}e(H, K)}{n} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}({}_{j}\mu_{Gm}^{H}(v_i) - {}_{j}\mu_{Gm}^{K}(v_i))^2} .$$

Definition 5.14. Let H and K be two subgraphs of G in a finite G_m-closure space (G, F_{Gm}). The relative j-Euclidean norm between H and K for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$ is define by

$$_{j}\varepsilon^2(H, K) = \frac{_{j}e^2(H, K)}{n} = \frac{1}{n}\sum_{i=1}^{n}({}_{j}\mu_{Gm}^{H}(v_i) - {}_{j}\mu_{Gm}^{K}(v_i))^2 .$$

Example 5.9. In Example (4.3)., we get
$_{\alpha}H^f = \{(v_1, 1/3), (v_2, 1/3), (v_3, 1/2), (v_4, 0)\}$, and
$_{\alpha}K^f = \{(v_1, 1), (v_2, 1), (v_3, 3/4), (v_4, 0)\}$. Then
$_{\alpha}\varepsilon(H, K) = \frac{0.975}{2} = 0.488$, and
$_{\alpha}\varepsilon^2(H, K) = \frac{(0.975)^2}{4} = 0.238$.

6. Conclusions

In this paper, we used G_m-closure space concepts to introduce definitions to near rough, near exact and near fuzzy graphs. We generalize near rough graphs in the frameworks of topological spaces. We believe such generalization will be useful in digital topology [22] as well as biomathematics [26]. The topological applications which introduced help for measuring near exactness and near roughness of graphs. Our approach is to topologize information systems. We connect near rough graphs, topological spaces, near rough membership function, and near fuzzy graphs.

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