Covering Space and Van Kampen theory methods of Fundamental Group

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Abstract

This paper examines two methods of computing fundamental group which are covering space method and the Van Kampen theory method. Van Kampen theory method is more analytical than the cover space method; the idea is used to solve a geometrical problem of global nature by first reducing it to homotopy theory problem which in turn reduces to an algebraic problem and solves as such. In 2001, H. Fausk et al [1] and Hu [2] showed the isomorphism between left and right adjoint theory and its application to homotopy categories, bearing in mind that homotopy group are higher dimension of fundamental group. In this paper detail explanations of fundamental group and homotopy group will be given. Various definitions and explanation of concepts, which are directly or indirectly related, will also be considered with illustration on how fundamental group can be calculated. This paper also reviews that in principle any space that can be broken up into pieces can have its fundamental group described by generators and relations via Van Kampen's theorem

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1. Introduction

The theory of fundamental groups and covering spaces is one of the few parts of algebraic topology that has probably reached definitive form, and it has not be generally treated in many sources. This review paper will explore its rudiments in actual sense.

In algebraic topology, homotopy theory is the study of homotopy groups, more generally of the category of topological spaces and homotopy classes of continuous mapping at an intuitive level, a homotopy class is a connected component of a function space, while homotopy group is said to be a higher dimension of fundamental group. Fundamental group is denoted as π(X, x) which consists of all equivalence classes of loops based at x and the product operation between them.

May J.P [3,4] defined topological space X as a set in which there is a notion of nearness of points, given a collection of open subsets of X which is closed under finite intersections and arbitrary unions. It is then suffice to imagine that metric spaces connotes open sets that are the arbitrary unions of finite intersections of neighbourhoods

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$U\varepsilon(x) = \{y|d(x, y) < \varepsilon\}$.
A function p: X $-\!\!\rightarrow$ Y is continuous if it takes nearby points to nearby points, p−1(U) is open if U is open. If X and Y are metric spaces, this means that, for any $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $p(U\delta(x)) \subset U\varepsilon(p(x))$. Algebraic topology assigns discrete algebraic invariants to topological spaces and continuous maps.

1.1 General topology

Let x be a non-empty set. A class τ of subset of x is a topology on x (point topology) iff τ satisfied the following axioms:
[01] X and $\emptyset$ belong to τ i.e., X, $\emptyset \in \tau$.
[02] The arbitrary union of any number of sets in τ belongs to τ.
[03] The finite intersection of any two sets in τ belongs to τ.
Therefore the pair (X,τ) is called a topological space.

1.2 Category

A category C consists of the following:
(a) A class of objects (family of set)
(b) For every ordered pair of objects A and B, a set Mov (A, B) of "Morphisms"
Iff f $\in$ Mov (A, B) we write: F: A $\rightarrow$ B or A $\xrightarrow{f}$ B
(c) For every ordered triple of objects A, B and C, a function is associated to a pair of morphisms f: A $\rightarrow$ B and g: B $\rightarrow$ C their "composite"
g $\circ$ f : A $\rightarrow$ C i.e. f $\in$ Mov(A,B), g $\in$ Mov(B,C) then g $\circ$ f $\in$ Mov(A,C).

1.3 Covariant Functor

Let C and D be categories respectively, a covariant functor is a map τ: C$\rightarrow$ D consisting of object function which assigns to every object A $\in$ C, an object T (A) $\in$ D and morphism function which assigns to every
f $\in$ Mov (A, B) $\in$ C a morphism T (f) $\in$ Mov (T(A), T(B)) Such that:
1. T (1A) = 1T (A) identity goes to identity
2. T (g $\circ$ f) = T (g) · T (f) composite goes to composite.

1.4 Contravariant functor

If C and D be categories respectively, a Contravariant functor is a map δ : C $\rightarrow$ D consist of an object which assign to every object A $\in$ C an object S(A) $\in$ D and a morphism function which assigns to every
f $\in$ Mov (A, B) $\in$ C a morphism S(f) $\in$ Mov(S(B), S(A)) in D Such that:
1. S (1A) = 1S (A) identity goes to identity
2. S (g $\circ$ f) = S (f) · S (g) composites goes to composite.

1.5 Exact sequence

An exact sequence consists of family Aq, q $\in$ Z of algebraic structure together with morphisms fq : Aq $\rightarrow$ Aq+1 such that we have a long sequence:
$\cdots Aq-1 \xrightarrow{fq-1} Aq \xrightarrow{fq} Aq+1 \xrightarrow{fq+1} Aq+2 \xrightarrow{fq+2} \cdots$
which is exact at every point of the sequence i.e. Im fq = kerfq+1 $\forall$ q.

1.6 Homotopy

Let X, Y $\in$ C, denoted by Y^* = {f: X $\rightarrow$ Y/f is a map} which is continuous f, g $\in$ Y^* are said to be homotopic if there exist a map
f: X × I $\rightarrow$ Y, I = [0, 1] the unit interval such that:
F (x, 0) = f (x) $\forall$ x $\in$ X
F (x, 1) = g (x) $\forall$ x $\in$ X;
then F is said to be homotopy written as f $\sim$ g (f is homotopic to g).

1.7 Definition of Arc or Path

A path in a topological space X is a continuous map of some closed interval into X i.e. f $\in X^I \Rightarrow$ f: I $\rightarrow$ X such that: x_0 = f(0) to x_1 = f(1). x_0 and x1 are initial and terminal point respectively.

Let X be a space and X1 andX2 are paths in X respectively such that
X1(1) = X2(0). Then the composite path X1 · X2 is given by

$$X1 \cdot X2 = \begin{cases} X_{1(2t)} & if\ 0 \le t \le \frac{1}{2} \\ X_{2(2t-1)} & if\ \frac{1}{2} \le t \le 1 \end{cases}$$

A space X is called arcwise connected or Pathwise connected if any two Points of X can be joined by an arc.

The path components of X are the maximal arcwise connected subsets of X (i.e. ordinary components of X).

If the map f: I $\rightarrow$ X is a path such that f (0) = f (1) is called a loop which is based at a point $x \in$ X.

1.8 Retract and Deformation Retract

A is said to be a retract of X if the identity map of A, 1A can be extended
to map r: X $\rightarrow$ A i.e. A $\xrightarrow{i}$ X $\xrightarrow{r}$ A.

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Let A ⊂ B ⊂ X, A is called a deformation retract of B over X if 1B $\sim$ a
Retract r: B → A.

1.9 Covering Spaces

Let X be a topological space, a covering space is a space $\tilde{X}$ and a map
$\pi : \tilde{X} \to$ X such that:
(1) π Is onto
(2) $\forall$ x ∈ X ∃ a neighbourhood V of □: π^{-1}(V) is disjoint union of Open
Sets each f which is mapped homeomorphically onto V by π where X is the base space and $\tilde{X}$ is the total space.

Examples of covering space are:
(1) $\pi_n : S' \to S'$ given by $\pi_n(Z) = Z^n$ covering space of n fold covering.
(2) : $R^n \to S' \times \cdot \cdot \cdot \times S'$ given by π(X1. . . Xn) = $(e^{2\pi ix} \ldots e^{2\pi ixn})$
(3) : [0, 1] × $R' \to$ [0, 1] × S'
π(s, t) = (s,$e^{2\pi it}$)
[0, 1] × S^{-1} Identity with {(x, y) ∈ R2|1 $\leq x^2 + y^2 \leq$ 4}
If: $\tilde{X}$ → X is a covering space such that $\tilde{X}$ is simply connected then the Covering space is called a universal covering space.

A connected space X is a space which is pathwise connected and whose fundamental group is trivial i.e. π_1(X) = 0.

1.10 Fundamental group

The class of map (homotopy class of map) $\pi(X,x_0)$ is referred to as the fundamental group for x ∈ X. It is the set of all the loops based at x_o which
For a group $\pi(X,x_0)$ = [I, (X,x_0)] {0, 1}.
If P: I → X then P (0) = P (1) it is a loop.

2.0 Calculating Fundamental Groups

Lewis et al [5] and May J.P [4] have supported that fundamental group can be analyzed and calculated by using two approaches:

2.1 Conversion of fundamental group problems to algebraic problems and solve as such, it is often achieved by putting algebraic structure on sets of homotopy.

Considering two methods of calculating fundamental group, the first method which is the covering spaces method is quite geometric and connections between the spaces is not necessary because it allow working based on intuition to the answer. The second method is the Van Kampen theorem which is analytical and some what used to show that the space is the map e: $R' \to S'$ given by e (t) = $e^{2\pi it}$ e is periodic of period 1.We think of a spiral connected of this space into a circle.

For the calculation of fundamental group there is need to relate it with the structure of covering spaces with the path lifting property.

Path lifting property
Given P: I → X and a ∈ $\tilde{X}$ such that:
π(a) = P (0).
There is a unique Path $\tilde{P}$: I → $\hat{S}$ such that $\pi\tilde{P}$ = P and P (0) = a.

Example using covering space
Homomorphism $\mathbb{Z} \to \pi_1$ (S^1,1): n ↦ α^n is an isomorphism the formula n ↦ α^n determines a homomorphism $\mathbb{Z} \to \pi_1$ (S^1,1) then show that loop s : I → S^1 starting at 1 is homotopic to S_n if the path $\tilde{S}$: I → $\mathbb{R}$ covering s and starting at 0 ∈ $\mathbb{R}$ end s at x ∈ $\mathbb{R}$.
Also that S_n is null homotopic if n = 0.

Solution

The map $\mathbb{Z} \to \pi_1$ (S^1,1) is well defined homomorphism, by map s: I → S^1 it is an epimorphism and by map $\tilde{S}$: I → $\mathbb{R}$ it is a monomorphism, therefore it is an isomorphism.
If n ↦ α^n and K ↦ α^K, then n + k ↦ $\alpha^{n+K} = \alpha^n . \alpha^K$
$\mathbb{R}$ is connected, the paths $\hat{S}$ and $\hat{S}^n$ are homotopic therefore the the path S and S_n are homotopic. [S] = [S_n] =α^n. But if n ≠ 0 then the path is not a loop and loop S_n is not null – homotopic.

This method is not analytical enough; therefore, we describe a tool for calculating $\pi(X,x_0)$.
Assume X = X1 ∪ X2 and X1 ∩ X2 ≠ ∅ then choosing x_0 ∈ X1 ∩ X2
We have:

$i_1 : \pi(X1 \cap X2, x_0) \rightarrow \pi(X1, x_0)$ and
$i_2 : \pi(X1 \cap X2, x_0) \rightarrow \pi(X2, x_0)$ which is homorphisms, making a general group construction.
Let G1 and G2 be groups f1 : G → G1 and f2 : G → G2 homorphism. The amalgamation of G1 and G2 over G is the smallest group generated by G1 and G2 with f1 (x) = f2 (x) for $x \in G$.
If F is the free group generated by G1 ∪ G2 then:
$x \cdot y$ is three products in F
F is of the form $x_1^{\epsilon_1} \ \ldots \ x_k^{\epsilon_n}, \epsilon_1 = \pm 1$ and $X \in G1 \cup G2$.

The Van Kampen theorem allows the calculation of $\pi(X, x_0)$ provided π(X1), π(X2) and π(X1 ∩ X2) are known.

2.1 Van Kampen Theory

The statement and prove of the theorem Van Kampen theorem are as follows:
As X1 and X2 are connected space open subsets of X such that X = X1 ∪ X2 and X1 ∩ X2 = $\emptyset$ and are connected, choosing a base points $x_0 \in X1 \cup X2$ for all $\pi(X, x_0)$ under consideration.

Theorem 2.1.1: Let H be any group and P1, P2, P3 are the homomorphism

Such that:

$$\begin{tikzcd}
 & \pi(X_1) \arrow[dr, "\rho 1"] & \\
\pi(X_1 \cap X_2) \arrow[ur, "\emptyset_1"] \arrow[dr, "\emptyset_2"'] \arrow[rr, "\rho 3"] & & H \\
 & \pi(X_2) \arrow[ur, "\rho 2"'] &
\end{tikzcd}$$

There exists a unique homomorphism $\sigma : \pi(X) \rightarrow H$ such that the diagrams is commutative

$$\begin{tikzcd}
 & \pi(X) \arrow[d, "\sigma"] \\
\pi(X_1) \arrow[ur, "\emptyset_1"] \arrow[r, "\rho 1"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi(X) \arrow[d, "\sigma"] \\
\pi(X_2) \arrow[ur, "\emptyset_2"] \arrow[r, "\rho 2"'] & H
\end{tikzcd}$$

$$\begin{tikzcd}
 & \pi(X) \arrow[d, "\sigma"] \\
\pi(X_1 \cap X_2) \arrow[ur, "\emptyset_3"] \arrow[r, "\rho 3"'] & H
\end{tikzcd}$$

The prove of this theorem is to show that π(X) is a characterized up to Isomorphism by the theorem i.e
$\sigma : \pi(X) \to H$

Proof:
To achieve this we show that it is
Homorphism, it is one-on-one and onto.
Assume $x_1; x_2 \in H$.
Then $\sigma : \pi(X) \to H$ is well defined,
then $\sigma((x_1)(x_2)) = \sigma(x_1) : \sigma(x_2)$ by definition
$\sigma((x_1)(x_2)) = \sigma((x_1, x_2))$
$= (\alpha^{-1}x_1, x_2\alpha)$
$= ((\alpha^{-1}x_1\alpha))((\alpha^{-1}x_2\alpha))$
$= \sigma(x_1) \cdot \sigma(x_2)$
Therefore, it is homorphism.

Assume $\sigma(x_1) = \sigma(x_2)$
If $x_1 = x_2$ then it is 1 – 1.
Given $(\alpha^{-1}x_1\alpha) = (\alpha^{-1}x_2\alpha)$
$\Rightarrow \alpha^{-1}x_1\alpha \simeq \alpha^{-1}x_2\alpha$
$\Rightarrow x_1 = x_2$ hence it is 1 − 1

Let $x_2 \in H$ to determine $x_1 \in \pi(x)$ such that $\sigma(x_1) = x_2$.
Consider $x_1 = \alpha x_2\alpha^{-1}$ a loop at x_0. Then
$\sigma(x_1) = \sigma(\alpha x_2\alpha^{-1})$
$= (\alpha^{-1}(\alpha x_2\alpha^{-1})\alpha)$
$= 1 \cdot x_2 \cdot 1$
$= x_2$.
Hence it is isomorphism.

The most general version of Van Kampen theorem consist of covering Space X by any number of open sets which is not just two open sets. This Open set must be arcwise connected; also the intersection of any finite number must be arcwise connected containing the base point.

Illustration of Van Kampen method
Let X be a space X = A [B, A \ B = $\{x_0\}$ and A and B are each Homeomorphic to circle S0, X may be visualized as follows
Let X1 = A1, X2 = B to determine the
Structure $\pi(x)$ but A and B is not open.
Let a $\in$ A and b $\in$ B: a $\neq x_0$ and b $\neq x_0$.
Let X1 = X − {b} and X2 = X − {a}, X1 and X2 are homeomorphic to a
Circle X1 \ X2 = X − {a, b} is contractible.
Hence they are simply connected.

Thus $\pi(x)$ is a free product of the group $\pi(x_1)$ and $\pi(x_2)$.
Thus π(A) and π(B) are infinite cyclic group.

2.2 Interpolation of Fundamental Group and Covering space

Suppose Z is a space, and * a point of Z. We define π_1(Z,*) as homotopy classes of maps f:[0,1] --> Z, such that f(0) = f(1) = *.

Shmuel [6] proved that the boundary conditions are absolutely critical for getting a nontrivial theory. π_1 (Z,*) is a group using concatenation of paths; the constant path is the identity and "going backwards is the inverse. π_1 (Z,*) is referred to as the fundamental group of Z. (If Z is path connected, the choice of * is irrelevant.

Example: If Z is the circle S^1 = {u in **C** | |u| = 1}, we can define a map π_1 (S^1,1) -> **Z** (the integers) by sending a map f to
(log(f)(1) – log(f)(0))/2 πi.

Definition. A map p: A -> B is a covering space, if:around each point b in B, there is a neighborhood N of b, so that p^{-1} (N) is a disjoint union of sets Ai each of which is mapped homeomorphically onto N by p.
The map exp: **R** -> S^1 considered before is a good example.

Examples: The 2-sphere S^2 is simply connected. The projective plane **RP**^{2} has fundamental group **Z**/2**Z** since it is the quotient of S^2 by making the identifications x = -x. The projection map is a covering map, and the group of covering transformations is just **Z**/2**Z** = {id, x -> -x}. The nontrivial element in the fundamental group of **RP2** can be thought of
as the quotient of a great chord on S2 that connects the north pole to the south pole.

2.3 Computation of Fundamental Group.

Fundamental group discussed earlier has two popularilities; the first being its connection to covering space theory. The second is that it is quite computable that is Van Kampen theory
Example: If X is contractible then the fundamental group is trivial.
Example: If one sees the universal cover and group of deck transformations, then one also knows the fundamental group.

The practical tool of this computation is Van Kampen's theorem.

Van Kampen's theorem. Let Z denote the union of A and B, and X denote their intersection. If A, B, and X are all connected (and nonempty), and then π_1 (Z,x) is generated by π_1 (A,x) and π_1 (B,x). The only relations among the elements of π_1 (A,x) and π_1 (B,x) are the ones forced by the fact that the elements of π_1 (C,x) can be thought of as elements of both of these groups.

Examples.
1. If A and B are simply connected, and their intersection is connected, then their union is simply connected.
2. If X is simply connected, then π_1 (Z,x) is the free product
π_1 (A,x)* π_1 (B,x). The elements of the free products are just finite strings of elements of π_1 (A,x) and π_1 (B,x), and one multiplies strings by concatenting them, ignoring the identity, and combining contiguous elements of the same group.

3. These groups can be tricky if π_1 (X,x) is nontrivial. The group described is called a free product with amalgamation and is denoted by π_1(A,x)* π_1 (X,x) π_1 (B,x).

Interpretation of this is that the elements of this look like when the induced maps of π_1 (X,x) into the other two pieces are injective, but without this it can get complicated. As a simple example suppose that X is a circle and that π_1 (A,x) = **Z**/2**Z** and π_1 (B,x) = **Z**/3**Z**, so that the induced homomorphisms are the obvious surjections. Hence, space is established and that Van Kampen's theorem tells us that Z is simply connected.

The fundamental groups of both A and B are generated by that of the circle, i.e. there is one generator, say g. From A we learn that g^2 = e and from B we learn that g^3 = e. So in the amalgamated free product (i.e. π_1 (Z,x)) g = e, so the who group vanishes.

Conclusion

Fundamental group have been treated geometrically, it was formulated in a simple way with algebraic convention and some of its concepts and theorems such as: concept relating homotopy maps with homotopy group were briefly reviewed. The higher dimension of this fundamental group is applicable to spaces such as: Real projective spaces, Complex projective spaces, Moore space M (π, n) etc., but for this paper calculation of fundamental group was only discussed because this is necessary before the application. In principle any space that can be broken up into pieces can have its fundamental group described by generators and relations via Van Kampen's theorem and then calculated appropriately.

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