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Existence of Solutions of System of Generalized Vector Quasi-Equilibrium Problems in Product FC-spaces

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Abstract

In this paper, we introduce four new types of the system of generalized vector quasi-equilibrium problems in finitely continuous topological spaces (in short, FC-spaces). By a maximal element theorem in product FC-spaces, we prove the existence of solutions for such kinds of system of generalized vector quasi-equilibrium problems. These theorems improve, unify many important result in recent literature.

Keywords: generalized vector quasi-equilibrium; FC-space; maximal element

1 Introduction

In recent years, the equilibrium problem with vector-valued functions and set-valued maps have been studied in [1-3] and the references therein. Very recently, the system of vector quasi-equilibrium problems, i.e., a family of quasi-equilibrium problems for vector-valued bifunctions defined on a product set, was introduced by Ansari et al.[4] with applications in Debreu type equilibrium problem for vector-valued functions. This problem was extensively investigated and generalized in [5-6] and existence results of a solution have been proved.

Let I be a finite or a infinite index set. For each $i \in I$, let Z_i be Haudorff topological space and $(X_i, \{\varphi_{N_i}\})_{i\in I}$, $(Y_i, \{\varphi_{M_i}\})_{i\in I}$ be FC-spaces. We denote by 2^X and $\langle X\rangle$the family of all subsets of X and the family of all nonempty finite subsets of X respectively. Let Δ_n be the standard n-dimensional simplex with vertices $e_0, \dots, e_n$. If J is a nonempty subset of $\{0, 1, \dots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. Let $D_i : X = \prod_{i\in I} X_i \to 2^{X_i}, T_i : X = \prod_{i\in I} X_i \to 2^{Y_i}$ and $C_i : X = \prod_{i\in I} X_i \to 2^{Z_i}$ be set-valued maps with nonempty values, $C_i(x)$ is a proper closed convex cone with apex at the origin and $intC_i(x) \neq \emptyset$. Let $F_i : X \times Y \times X_i \to 2^{Z_i}$ be a set-valued map, where $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$. Let $\pi_i : X \to X_i$, $\theta_i : Y \to Y_i$ be projective mappings from X to X_i and Y to Y_i respectively.

In this paper, we study the following classes of the system of the generalized vector quasi-equilibrium problems:

(1) find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})$ and $F_i(\bar{x}, \bar{y}, z_i) \subset C_i(\bar{x})$ for all $z_i \in D_i(\bar{x})$.

(2) find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})$ and $F_i(\bar{x}, \bar{y}, z_i) \cap C_i(\bar{x}) \neq \emptyset$ for all $z_i \in D_i(\bar{x})$.

(3) find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})$ and $F_i(\bar{x}, \bar{y}, z_i) \cap (-intC_i(\bar{x})) = \emptyset$ for all $z_i \in D_i(\bar{x})$.

(4) find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})$ and $F_i(\bar{x}, \bar{y}, z_i) \nsubseteq (-intC_i(\bar{x}))$ for all $z_i \in D_i(\bar{x})$.

Recently Lin et al. [7] studied the following problems:

(i) find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for all $i \in I, \bar{x}_i \in clS_i(\bar{x}), F_i(t_i, \bar{x}, y_i) \subset C_i(\bar{x})$ for all $y_i \in S_i(\bar{x})$, and for all $t_i \in T_i(\bar{x})$

(ii) find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for all $i \in I, \bar{x}_i \in clS_i(\bar{x})$ and for each $y_i \in S_i(\bar{x})$, there exists $t_i \in T_i(\bar{x})$ such that $F_i(t_i, \bar{x}, y_i) \cap C_i(\bar{x}) \neq \emptyset$.

(iii) find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for each $i \in I, \bar{x}_i \in clS_i(\bar{x}), F_i(t_i, \bar{x}, y_i) \cap (-intC_i(\bar{x})) = \emptyset$ for all $y_i \in S_i(\bar{x})$, and for all $t_i \in T_i(\bar{x})$

(iv) find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for all $i \in I, \bar{x}_i \in clS_i(\bar{x})$ and for each $y_i \in S_i(\bar{x})$, there exists

$t_i \in T_i(\bar{x})$ such that $F_i(t_i, \bar{x}, y_i) \nsubseteq (-intC_i(\bar{x}))$.

where Z_i is a Hausdorff t.v.s, X_i and D_i are nonempty subsets of two Hausdorff t.v.s E_i and V_i respectively. $S_i : X \to 2^{X_i}, T_i : X \to 2^{D_i}, C_i : X \to 2^{Z_i}$ and $F_i : D_i \times X \times X_i \to 2^{Z_i}$ are multivalued maps.

Lin et al. [8] also studied the following problems:

(i') find $\hat{x}, \hat{y} \in X$ such that for each $i \in I, \hat{x_i} \in \bar{S}_i(\hat{x}), \hat{y_i} \in \bar{T}_i(\hat{x})$ and $f_i(\hat{x}, \hat{y}, u_i) \subset C_i(\hat{x})$ for all $u_i \in S_i(\hat{x})$.

(ii') find $\hat{x}, \hat{y} \in X$ such that for each $i \in I, \hat{x_i} \in \bar{S}_i(\hat{x}), \hat{y_i} \in \bar{T}_i(\hat{x})$ and $f_i(\hat{x}, \hat{y}, u_i) \cap C_i(\hat{x}) = \emptyset$ for all $u_i \in S_i(\hat{x})$.

(iii') find $\hat{x}, \hat{y} \in X$ such that for each $i \in I, \hat{x_i} \in \bar{S}_i(\hat{x}), \hat{y_i} \in \bar{T}_i(\hat{x})$ and $f_i(\hat{x}, \hat{y}, u_i) \cap (-intC_i(\hat{x})) = \emptyset$ for all $u_i \in S_i(\hat{x})$.

(iv') find $\hat{x}, \hat{y} \in X$ such that for each $i \in I, \hat{x_i} \in \bar{S}_i(\hat{x}), \hat{y_i} \in \bar{T}_i(\hat{x})$ and $f_i(\hat{x}, \hat{y}, u_i) \nsubseteq (-intC_i(\hat{x}))$ for all $u_i \in S_i(\hat{x})$.

where $f_i : X \times X \times X_i \to 2^{Z_i}, T_i : X \to 2^{X_i}, C_i : X \to 2^{Z_i}$ and $S_i : X \to 2^{X_i}$ are multivalued maps and $\hat{y_i} \in \bar{T}_i(\hat{x})$ means that $(\hat{x}, \hat{y_i}) \in GrT_i$

Our problems, our approaches and results are different from [7-8].

2 Preliminaries

The following notions was introduced by Ding in [9-12]

Definition 2.1. Let X and Y be topological spaces. A subset A of X is said to be compactly open (respectively, compactly closed) if for each nonempty compact subset K of X, $A \cap K$ is open (respectively, closed) in K.

Definition 2.2. The compact interior and the compact closure of A are defined by

$$\begin{aligned} cintA &= \cup\{B \subset X : B \subset A \text{ and } B \\ &\quad \text{is compactly open in} X\}, \\ cclA &= \cap\{B \subset X : A \subset B \text{ and } B \\ &\quad \text{is compactly closed}\} \end{aligned}$$

Clearly, we have $X \setminus cintA = ccl(x \setminus A)$ and $X \setminus cclA = cint(x \setminus A)$. For any compact subset K of X, we have $cintA \cap K = int_K(A \cap K)$ and $cclA \cap K = cl_K(A \cap K)$.

Definition 2.3. A set-valued mapping $T : X \to 2^Y$ is said to be transfer compactly open-valued if for $x \in X$ and for each compact subset K of $Y, y \in T(x) \cap K$ implies that there exist $x' \in X$ such that $y \in int_K(T(x') \cap K)$.

Definition 2.4. $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exist a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of Y if for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \cap D, \varphi_N(\Delta_k) \subset D$ where $\Delta_k = co(\{e_{i_j} : j = 0, \ldots, k\})$.

Clearly, each FC-subspace D of a FC-space $(Y, \{\varphi_N\})$ is also a FC-space.

Lemma 2.1. Let I be any index set. For each $i \in I$, let $(Y_i, \{\varphi_{N_i}\})$ be a FC-space. Let $Y = \prod_{i \in I} Y_i$ and $\varphi_N = \prod_{i \in I} \varphi_{N_i}$. Then $(Y, \{\varphi_N\})$ is also a FC-space.

Theorem 2.1. **[13]** Let E_1, E_2 and Z be real t.v.s., X and Y be nonempty subset of E_1 and E_2, respectively. Let $F : X \times Y \to 2^Z, S : X \to 2^Y$ be multivalued maps.

(i) if both S and F are l.s.c., then $T : X \to 2^Z$ defined by $T(x) = \cup_{y \in S(x)} F(x, y)$ is l.s.c. on X;

(ii) if both F and S are u.s.c., with compact values, then T is an u.s.c. multivalued map with compact values.

Theorem 2.2. **[14]** Let X and Y be topological spaces, $F : X \to 2^Y$ be a multivalued map.

(i) if $F : X \to 2^Y$ is an u.s.c. multivalued map with closed values, then F is closed.

(ii) if F is compact and $F : X \to 2^Y$ is an u.s.c. multivalued map with compact values, then $F(X)$ is compact.

Proposition 2.2. **[15]** Let X and Y be topological spaces, $F : X \to 2^Y$ be a multivalued map. Then F is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there is net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ for every α and y_α converging to y.

We shall use the following maximal theorem due to Ding [9].

Theorem 2.3. Let I be an any index set. For each $i \in I$, let $(X_i, \{\varphi_{N_i}\})$ be a FC-space and let $X = \prod_{i \in I} X_i$ such that $(X, \{\varphi_N\})$ is a FC-space defined as in lemma 2.1. For each $i \in I$, let $A_i : X \to 2_i^X$ such that

(i) for each $x \in X$, $A_i(x)$ is a FC-subspace of

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X_i,

(ii)for each $x \in x, x_i = \pi_i(x) \notin A_i(x)$ and $A_i^{-1} : X_i \to 2^X$ is transfer compactly open-valued.

(iii) for each $x \in X, I(x) = \{i \in I : A_i(x) \neq \emptyset\}$ is finite.

(iv) there exists a compact subset K of X and for each $i \in I$ and $N_i \in \langle X_i \rangle$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i such that for each $x \in X \setminus K$, there exists $y \in L_N = \prod_{i \in I} L_{N_i}$ such that for each $i \in I(x), x \in cintA_i^{-1}(\pi_i(y))$.

Then there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

3 Existence theorems

Some existence results of a solution for the four types of system of generalized vector quasi-equilibrium problems are shown.

Theorem 3.1. Let I be an any index set. For each $i \in I$, let $(X_i, \{\varphi_{N_i}\})$ and $(Y_i, \{\varphi_{M_i}\})$ be FC-spaces, let $D_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be set-valued maps. For each $i \in I$, assume that

(i) for each $x \in X, D_i(x)$ and $T_i(x)$ are nonempty FC-subspaces of X_i and Y_i respectively.

(ii) for all $(x, y) \in X \times Y$, the set $\{z_i \in X_i : F_i(x, y, z_i) \nsubseteq C_i(x)\}$ is nonempty FC-subspace of X_i.

(iii) for all $(x, y) \in X \times Y$ and each $x_i = \pi_i(x)$, we have $F_i(x, y, x_i) \subset C_i(x)$.

(iv)for each $i \in I$, $F_i : X \times Y \times X_i \to 2^{Z_i}$ is lower semicontinuous on $X \times Y$ and $C_i : X \to 2^{Z_i}$ is upper semicontinuous with closed values.

(v) for each $y_i \in X_i$ and $a_i \in Y_i$, $D_i^{-1}(y_i), T_i^{-1}(a_i)$ are compactly open.

(vi) the set $W_i = \{(x, y) \in X \times Y : x_i = \pi_i(x) \in D_i(x)$ and $y_i = \theta_i(y) \in T_i(x)\}$ is compactly closed;

(vii) for each $(x, y) \in X \times Y$, there exists $z_i \in D_i(x)$ such that $I(x, y) = \{i \in I : F_i(x, y, z_i) \nsubseteq C_i(x)\}$ is finite.

(viii) there exist nonempty and compact subsets $K \subseteq X$ and $N \subseteq Y$ and for each $i \in I$ and $B_i \subset \langle X_i \rangle, A_i \subset \langle Y_i \rangle$, there exist compact FC-subspaces L_{B_i} of $\langle X_i \rangle$ and L_{A_i} of $\langle Y_i \rangle$ containing B_i and A_i respectively, such that for each $(x, y) \in (X \times Y) \setminus (K \times N)$, there exists $(u, v) \in L_B \times L_A$, where $L_B = \prod_{i \in I} L_{B_i}$ and $L_A = \prod_{i \in I} L_{A_i}$, such that for each $i \in I(x, y), F_i(x, y, \pi_i(u)) \nsubseteq C_i(x)$ and $\theta_i(v) \in T_i(x)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x_i} = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y_i} = \theta_i(\bar{y}) \in T_i(\bar{x}), F_i(\bar{x}, \bar{y}, z_i) \subset C_i(\bar{x})$ for all $z_i \in D_i(\bar{x})$

Proof. For each $i \in I$, let us define a set-valued map $P_i : X \times Y \to 2^{X_i}$ by

$$P_i(x, y) = \{z_i \in X_i : F_i(x, y, z_i) \nsubseteq C_i(x)\},$$

where $\forall (x, y) \in X \times Y$. Then, $P_i(x, y)$ is a FC-subspace of X_i. By condition (iii), we have $x_i = \pi_i(x) \notin P_i(x, y)$. By (iv) and Theorem 2.1 it follows that for each $z_i \in x_i, P_i^{-1}(z_i)$ is compactly open. Indeed, if $(x, y) \in X \times Y \setminus P_i^{-1}(z_i)$, then there exists a net $\{x^\alpha, y^\alpha\}$ in $X \times Y \setminus P_i^{-1}(z_i)$ such that $\{x^\alpha, y^\alpha\} \to (x, y) \in X \times Y$, and $F_i(x^\alpha, y^\alpha, z_i) \subset C_i(x^\alpha)$. Let $u_i \in F_i(x, y, z_i)$, by (iv) $(x, y) \to F_i(x, y, z_i)$ is l.s.c for each $z_i \in X_i$. By Proposition 2.2, there exists a net $\{u_i^\alpha\}$ in $F_i(x^\alpha, y^\alpha, z_i)$ such that $u_i^\alpha \to u_i$. Therefore $u_i^\alpha \in C_i(x^\alpha)$. Since $C_i : X \to 2_i^Z$ is an u.s.c multivalued map with closed values, it follows from Theorem 2.2 that C_i is a closed multivalued map. Therefore, $u_i \in C_i(x)$ and $F_i(x, y, z_i) \subset C_i(x)$. We saw that $(x, y) \in X \times Y$. Therefore, $(x, y) \in X \times Y \setminus P_i^{-1}(z_i)$ and $X \times Y \setminus P_i^{-1}(z_i)$ is closed for all $z_i \in X_i$. This shows that $P_i^{-1}(z_i)$ is open for all $z_i \in X_i$. Hence, $P_i^{-1}(z_i)$ is compactly open.

By lemma 2.1, $(X \times Y, \{\varphi_N\})$ is also a FC-space where $X \times Y = \prod_{i \in I}(X_i \times Y_i)$.

For each $i \in I$, we also define another set-valued map $S_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$S_i(x, y) = \begin{cases} [D_i(x) \times P_i(x, y)] \times T_i(x), (x, y) \in W_i; \\ D_i(x) \times T_i(x), (x, y) \notin W_i; \end{cases}$$

Then by (i) and $P_i(x, y)$ is a FC-subspace, for each $i \in I$ and for each $(x, y) \in X \times Y, S_i(x, y)$ is a FC-subspace of X_i and so the condition (i) of Theorem 2.3 is satisfied. By (iii) and the definition of W_i, we have $(x_i, y_i) = (\pi_i(x), \theta_i(y)) \notin S_i(x, y)$ for each $i \in I$ and for any $(x, y) \in X \times Y$. For each $i \in I$ and for any $(u_i, v_i) \in X_i \times Y_i$, we have

$$\begin{aligned} S_i^{-1}(u_i, v_i) \quad = \quad & [P^{-1}(u_i) \cap (D_i^{-1}(u_i) \times Y) \cap \\ & (T_i^{-1}(v_i) \times Y)] \cup [((X \times Y) \setminus W_i) \\ & \cap (D_i^{-1}(u_i) \times Y) \cap (T_i^{-1}(v_i) \times Y)] \end{aligned}$$

By the conditions (v) and (vi), $S_i^{-1}(u_i, v_i)$ is compactly open-valued and hence S_i^{-1} is transfer compactly open-valued on $X_i \times Y_i$. The condition (ii) of Theorem 2.3 is satisfied. The condition (vii) implies that the condition (iii) of Theorem 2.3 holds. Note that S_i^{-1} is compactly open-valued. From condition (viii), we have

$$\begin{aligned}(X \times Y) \setminus (K \times N) &\subset \cup\{S_i^{-1}(\pi_i(u), \theta_i(v)) : (u,v) \in L_N \times L_M\} \\ &= \cup\{cintS_i^{-1}(\pi_i(u), \theta_i(v)) : (u,v) \in L_N \times L_M\}\end{aligned}$$

and so the condition (iv) of Theorem 2.3 is satisfied. By Theorem 2.1, there exists $(\hat{x}, \hat{y}) \in X \times Y$ such that $S_i(\hat{x}, \hat{y}) = \emptyset$ for all $i \in I$. If $(\hat{x}, \hat{y}) \notin W_j$ for some $j \notin I$, then either $D_i(\hat{x}) = \emptyset$ or $T_i(\hat{x}) = \emptyset$ which contradicts the fact that $D_i(x)$ and $T_i(x)$ are both nonempty for each $x \in X$ and for any $i \in I$. Therefore, we have $(\hat{x}, \hat{y}) \in W_i$ for all $i \in I$, and hence for each $i \in I$, $\hat{x_i} = \pi_i(\hat{x}) \in D_i(\hat{x}), \hat{y_i} = \theta_i(\hat{y}) \in T_i(\hat{x})$ and $D_i(\hat{x}) \cap P_i(\hat{x}, \hat{y}) = \emptyset$, for all $i \in I$. Therefore, for all $i \in I$,

$$\hat{x_i} = \pi_i(\hat{x}) \in D_i(\hat{x}), \bar{y_i} = \theta_i(\hat{y}) \in T_i(\hat{x}),$$

$$F_i(\hat{x}, \hat{y}, z_i) \subset C_i(\hat{x}) \text{ for all } z_i \in D_i(\hat{x})$$

This completes the proof.

Following the same argument as Theorem 3.1, we can prove the following theorem.

Theorem 3.2. For each $i \in I$, assume that

(i) for each $x \in X, D_i(x), T_i(x)$ are nonempty FC-subspaces of X_i and Y_i respectively.

(ii) for all $(x,y) \in X \times Y$, the set $\{z_i \in X_i : F_i(x,y,z_i) \cap C_i(x) = \emptyset\}$ is nonempty FC-subspace of X_i.

(iii) for all $(x,y) \in X \times Y$ and each $x_i = \pi_i(x)$ we have $F_i(x,y,x_i) \cap C_i(x) \neq \emptyset$.

(iv) for each $i \in I, F_i : X \times Y \times X_i \to 2^{Z_i}$ is upper semicontinuous with compact values and $C_i : X \to 2^{Z_i}$ is upper semicontinuous.

(v) for each $y_i \in X_i$ and each $a_i \in Y_i$, $D_i^{-1}(y_i), T_i^{-1}(a_i)$ are compactly open.

(vi) the set $W_i = \{(x,y) \in X \times Y : x_i = \pi_i(x) \in D_i(x) \text{ and } y_i = \theta_i(y) \in T_i(x)\}$ is compact closed.

(vii) for each $(x,y) \in X \times Y$, there exists $z_i \in D_i(x)$ such that $I(x,y) = \{i \in I : F_i(x,y,z_i) \cap C_i(x) = \emptyset\}$ is finite.

(viii) there exist nonempty and compact subsets $K \subseteq X$ and $N \subseteq Y$ and for each $i \in I$ and $B_i \subset \langle X_i \rangle, A_i \subset \langle Y_i \rangle$, there exist compact FC-subspaces L_{B_i} of $\langle X_i \rangle$ and L_{A_i} of $\langle Y_i \rangle$ containing B_i and A_i respectively, such that for each $(x,y) \in X \times Y \setminus K \times N$, there exists $(u,v) \in L_B \times L_A$, where $L_B = \prod_{i \in I} L_{B_i}$ and $L_A = \prod_{i \in I} L_{A_i}$, such that for each $i \in I(x,y)$, $F_i(x,y,\pi_i(u)) \cap C_i(x) = \emptyset$ and $\theta_i(v) \in T_i(x)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x_i} = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y_i} = \theta_i(\bar{y}) \in T_i(\bar{x})$ and $F_i(\bar{x}, \bar{y}, z_i) \cap C_i(\bar{x}) \neq \emptyset$ for all $z_i \in D_i(\bar{x})$

Proof. Let $P_i : X \times Y \to 2^{X_i}$ by $P_i(x,y) = \{z_i \in X_i : F_i(x,y,z_i) \cap C_i(x) = \emptyset\}, \forall (x,y) \in X \times Y$.

Then, $p_i(x,y)$ is a FC-subspace of X_i. By condition (iii), we have $x_i = \pi_i(x) \notin P_i(x,y)$. By (ii) and Theorem 2.1 it follows that for each $z_i \in X_i, P_i^{-1}(z_i)$ is open. Indeed, if $(x,y) \in X \times Y \setminus P_i^{-1}(z_i)$, then there exists a net $\{x^\alpha, y^\alpha\} \in (X \times Y) \setminus P_i^{-1}(z_i)$, such that $\{x^\alpha, y^\alpha\} \to (x,y) \in X \times Y$ and $F_i(x^\alpha, y^\alpha, z_i) \cap C_i(x^\alpha) \neq \emptyset$. Let $u_i^\alpha \in F_i(x^\alpha, y^\alpha, z_i) \cap C_i(x^\alpha)$. By (iv) and Theorem 2.2 that for each $z_i \in X_i, (x,y) \to F_i(x,y,z_i)$ is an u.s.c multivalued map with compact values. It suffices to find a subset $\{u_i^{\alpha_\lambda}\}$ of $\{u_i^\alpha\}$, which converges to some $u_i \in F_i(x,y,z_i)$. Since for each $z_i \in X_i$, the multivalued map $(x,y) \mapsto F_i(x,y,z_i)$ and C_i are u.s.c with compact values, it follows from Theorem 2.2 that for each fixed $z_i \in X_i, (x,y) \mapsto F_i(x,y,z_i)$ and C_i are closed. Therefore, $(x,y) \in X \times Y$ and $u_i \in F_i(x,y,z_i) \cap C_i(x) \neq \emptyset$. This shows that $X \setminus P_i^{-1}(z_i)$ is closed for each $z_i \in X_i$. Hence $P_i^{-1}(z_i)$ is open for each $z_i \in X_i$.

By lemma 2.1, $(X \times Y, \{\varphi_N\})$ is also a FC-space where $X \times Y = \prod_{i \in I}(X_i \times Y_i)$.

For each $i \in I$, we also define another set-valued map $S_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$S_i(x,y) = \begin{cases} [D_i(x) \times P_i(x,y)] \times T_i(x), & (x,y) \in W_i; \\ D_i(x) \times T_i(x), (x,y) \notin W_i; \end{cases}$$

Then by (i) and $P_i(x,y)$ is a FC-subspace, for each $i \in I$ and for each $(x,y) \in X \times Y, S_i(x,y)$ is a FC-subspace of X_i and so the condition (i) of Theorem 2.3 is satisfied. By (b) and the definition of W_i, we have $(x_i, y_i) = (\pi_i(x), \theta_i(y)) \notin S_i(x,y)$ for each $i \in I$ and for any $(x,y) \in X \times Y$. For each $i \in I$ and for any $(u_i, v_i) \in X_i \times Y_i$, we have

$$\begin{aligned}S_i^{-1}(u_i, v_i) = &[P^{-1}(u_i) \cap (D_i^{-1}(u_i) \times Y) \cap \\ &(T_i^{-1}(v_i) \times Y)] \cup [((X \times Y) \setminus W_i) \\ &\cap (D_i^{-1}(u_i) \times Y) \cap (T_i^{-1}(v_i) \times Y)]\end{aligned}$$

By the conditions (v) and (vi), $S_i^{-1}(u_i, v_i)$ is compactly open-valued and hence S_i^{-1} is transfer compactly open-valued on $X_i \times Y_i$. The condition (ii) of Theorem 2.3 is satisfied. The condition (viii) implies that the condition (iii) of Theorem 2.3 holds. Note that S_i^{-1} is compactly open-valued. From condition (viii), we have

$$\begin{aligned}(X \times Y) \setminus (K \times N) \quad &\subset \quad \cup\{S_i^{-1}(\pi_i(u), \theta_i(v)) : \\ &\qquad (u, v) \in L_N \times L_M\} \\ &= \quad \cup\{cintS_i^{-1}(\pi_i(u), \theta_i(v)) : \\ &\qquad (u, v) \in L_N \times L_M\}\end{aligned}$$

and so the condition (iv) of Theorem 2.3 is satisfied. By Theorem 2.3, there exists $(\hat{x}, \hat{y}) \in X \times Y$ such that $S_i(\hat{x}, \hat{y}) = \emptyset$ for all $i \in I$. If $(\hat{x}, \hat{y}) \notin W_j$ for some $j \notin I$, then either $D_i(\hat{x}) = \emptyset$ or $T_i(\hat{x}) = \emptyset$ which contradicts the fact that $D_i(x)$ and $T_i(x)$ are both nonempty for each $x \in X$ and for any $i \in I$. Therefore, we have $(\hat{x}, \hat{y}) \in W_i$ for all $i \in I$, and hence for each $i \in I$, $\hat{x_i} = \pi_i(\hat{x}) \in D_i(\hat{x}), \hat{y_i} = \theta_i(\hat{y}) \in T_i(\hat{x})$ and $D_i(\hat{x}) \cap P_i(\hat{x}, \hat{y}) = \emptyset$, for all $i \in I$. Therefore, for all $i \in I$,

$$\hat{x_i} = \pi_i(\hat{x}) \in D_i(\hat{x}), \bar{y_i} = \theta_i(\hat{y}) \in T_i(\hat{x}),$$

$$F_i(\hat{x}, \hat{y}, z_i) \cap C_i(\hat{x}) \neq \emptyset \text{ for all } z_i \in D_i(\hat{x})$$

This completes the proof.

With the same argument as in Theorem 3.1 and Theorem 3.2, we can prove the following Theorem 3.3 and Theorem 3.4 respectively.

Theorem 3.3. For each $i \in I$, suppose that

(i) for each $x \in X, D_i(x), T_i(x)$ are nonempty FC-subspaces of X_i and Y_i respectively.

(ii) for all $(x, y) \in X \times Y$, the set $\{z_i \in X_i : F_i(x, y, z_i) \cap (-intC_i(x)) \neq \emptyset\}$ is nonempty FC-subspace of X_i.

(iii) for all $(x, y) \in X \times Y$ and each $x_i = \pi_i(x)$ we have $F_i(x, y, x_i) \cap (-intC_i(x)) = \emptyset$.

(iv)for each $i \in I$, $F_i : X \times Y \times X_i \to 2^{Z_i}$ is lower semicontinuous on $X \times Y$ and $C_i : X \to 2^{Z_i}$ is upper semicontinuous with closed values.

(v) for each $y_i \in X_i$ and $a_i \in Y_i$, $D_i^{-1}(y_i), T_i^{-1}(a_i)$ are compactly open.

(vi) the set $W_i = \{(x, y) \in X \times Y : x_i = \pi_i(x)$ and $y_i = \theta_i(y) \in T_i(x)\}$ is compactly closed

(vii) for each $(x, y) \in X \times Y$, there exists $z_i \in D_i(x)$ such that $I(x, y) = \{i \in I : F_i(x, y, z_i) \cap (-intC_i(x)) \neq \emptyset\}$ is finite.

(viii) there exist nonempty and compact subsets $K \subseteq X$ and $N \subseteq Y$ and for each $i \in I$ and $B_i \subset \langle X_i \rangle, A_i \subset \langle Y_i \rangle$, there exist compact FC-subspaces L_{B_i} of $\langle X_i \rangle$ and L_{A_i} of $\langle Y_i \rangle$ containing B_i and A_i respectively, such that for each $(x, y) \in (X \times Y) \setminus (K \times N)$, there exists $(u, v) \in L_B \times L_A$, where $L_B = \prod_{i \in I} L_{B_i}$ and $L_A = \prod_{i \in I} L_{A_i}$, such that for each $i \in I(x, y), F_i(x, y, \pi_i(u)) \cap (-intC_i(x)) \neq \emptyset$ and $\theta_i(v) \in T_i(x)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x_i} = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y_i} = \theta_i(\bar{y}) \in T_i(\bar{x}), F_i(\bar{x}, \bar{y}, z_i) \cap (-intC_i(x)) = \emptyset$ for all $z_i \in D_i(\bar{x})$

Theorem 3.4. For each $i \in I$, assume that

(i) for each $x \in X, D_i(x), T_i(x)$ are nonempty FC-subspaces of X_i and Y_i respectively.

(ii) for all $(x, y) \in X \times Y$, the set $\{z_i \in X_i : F_i(x, y, z_i) \subset (-intC_i(x))\}$ is nonempty FC-subspace of X_i.

(iii) for all $(x, y) \in X \times Y$ and each $x_i = \pi_i(x)$ we have $F_i(x, y, x_i) \nsubseteq (-intC_i(x))$.

(iv) for each $i \in I, F_i : X \times Y \times X_i \to 2^{Z_i}$ is upper semicontinuous with compact values and $C_i : X \to 2^{Z_i}$ is upper semicontinuous.

(v) for each $y_i \in X_i$ and each $a_i \in Y_i$, $D_i^{-1}(y_i), T_i^{-1}(a_i)$ are compactly open.

(vi) the set $W_i = \{(x, y) \in X \times Y : x_i = \pi_i(x) \in D_i(x)$ and $y_i = \theta_i(y) \in T_i(x)\}$ is compact closed.

(vii) for each $(x, y) \in X \times Y$, there exists $z_i \in D_i(x)$ such that $I(x, y) = \{i \in I : F_i(x, y, z_i) \subset (-intC_i(x))\}$ is finite.

(viii) there exist nonempty and compact subsets $K \subseteq X$ and $N \subseteq Y$ and for each $i \in I$ and $B_i \subset \langle X_i \rangle, A_i \subset \langle Y_i \rangle$, there exist compact FC-subspaces L_{B_i} of $\langle X_i \rangle$ and L_{A_i} of $< Y_i >$ containing B_i and A_i respectively, such that for each $(x, y) \in X \times Y \setminus K \times N$, there exists $(u, v) \in L_B \times L_A$, where $L_B = \prod_{i \in I} L_{B_i}$ and $L_A = \prod_{i \in I} L_{A_i}$, such that for each $i \in I(x, y)$, $F_i(x, y, \pi_i(u)) \subset (-intC_i(x))$ and $\theta_i(v) \in T_i(x)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x_i} = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y_i} = \theta_i(\bar{y}) \in T_i(\bar{x})$ and $F_i(\bar{x}, \bar{y}, z_i) \nsubseteq (-intC_i(\bar{x}))$ for all $z_i \in D_i(\bar{x})$.

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