Existence of Solutions of System of Generalized Vector Quasi-Equilibrium Problems in Product FC-spaces

Wenxin Zhu\(^1\), Yunyan Song\(^2\)

\(^1\)Basic Science Department, Tianjin Agricultural University
Tianjin, 300384, China

\(^2\)College of Science, Tianjin University of Technology
Tianjin, 300384, China

Abstract

In this paper, we introduce four new types of the system of generalized vector quasi-equilibrium problems in finitely continuous topological spaces (in short, FC-spaces). By a maximal element theorem in product FC-spaces, we prove the existence of solutions for such kinds of system of generalized vector quasi-equilibrium problems. These theorems improve, unify many important result in recent literature.

Keywords: generalized vector quasi-equilibrium; FC-space; maximal element

1 Introduction

In recent years, the equilibrium problem with vector-valued functions and set-valued maps have been studied in [1-3] and the references therein. Very recently, the system of vector quasi-equilibrium problems, i.e., a family of quasi-equilibrium problems for vector-valued bifunctions defined on a product set, was introduced by Ansari et al.[4] with applications in Debreu type equilibrium problem for vector-valued functions. This problem was extensively investigated and generalized in [5-6] and existence results of a solution have been proved.

Let I be a finite or an infinite index set. For each \(i \in I\), let \(Z_i\) be Hausdorff topological space and \((X_i, \{\varphi_{X_i}\}_{i \in I})\) be FC-space. We denote by \(2^X\) and \(\mathcal{X}\) the family of all subsets of \(X\) and the family of all nonempty finite subsets of \(X\) respectively. Let \(\Delta_n\) be the standard \(n\)-dimensional simplex with vertices \(e_0, \ldots, e_n\). If \(J\) is a nonempty subset of \([0, 1, \ldots, n]\), we denote by \(\Delta_J\) the convex hull of the vertices \(\{e_j : j \in J\}\). Let \(D_i : X = \prod_{i \in I} X_i \to 2^{X_i}, T_i : X = \prod_{i \in I} X_i \to 2^{X_i}\) and \(C_i : X = \prod_{i \in I} X_i \to 2^{X_i}\) be set-valued maps with nonempty values, \(C_i(x)\) is a proper closed convex cone with apex at the origin and \(\text{int}C_i(x) \neq \emptyset\). Let \(F_i : X \times Y \times X_i \to 2^{X_i}\) be a set-valued map, where \(x = \prod_{i \in I} x_i, Y = \prod_{i \in I} Y_i\). Let \(\pi_i : X \to X_i, \theta_i : Y \to Y_i\) be projective mappings from \(X\) to \(X_i\) and \(Y\) to \(Y_i\) respectively.

In this paper, we study the following classes of the system of the generalized vector quasi-equilibrium problems:

1. find \((\bar{x}, \bar{y}) \in X \times Y\) such that for each \(i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})\) and \(F_i(\bar{x}, \bar{y}, z_i) \subseteq C_i(\bar{x})\) for all \(z_i \in D_i(\bar{x})\).
2. find \((\bar{x}, \bar{y}) \in X \times Y\) such that for each \(i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})\) and \(F_i(\bar{x}, \bar{y}, z_i) \cap C_i(\bar{x}) \neq \emptyset\) for all \(z_i \in D_i(\bar{x})\).
3. find \((\bar{x}, \bar{y}) \in X \times Y\) such that for each \(i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})\) and \(F_i(\bar{x}, \bar{y}, z_i) \cap (\text{int}C_i(\bar{x})) = \emptyset\) for all \(z_i \in D_i(\bar{x})\).
4. find \((\bar{x}, \bar{y}) \in X \times Y\) such that for each \(i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})\) and \(F_i(\bar{x}, \bar{y}, z_i) \cap (\text{int}C_i(\bar{x})) \neq \emptyset\) for all \(z_i \in D_i(\bar{x})\).

Recently Lin et al. [7] studied the following problems:

\(\text{(i)}\) find \(\bar{x} = (\bar{x}_i)_{i \in I} \in X\) such that for all \(i \in I, \bar{x}_i \in \text{cl}S_i(\bar{x}), F_i(t_i, \bar{x}, y_i) \subseteq C_i(\bar{x})\) for all \(y_i \in S_i(\bar{x})\), and for all \(t_i \in T_i(\bar{x})\).

\(\text{(ii)}\) find \(\bar{x} = (\bar{x}_i)_{i \in I} \in X\) such that for all \(i \in I, \bar{x}_i \in \text{cl}S_i(\bar{x})\) and for each \(y_i \in S_i(\bar{x})\), there exists \(t_i \in T_i(\bar{x})\) such that \(F_i(t_i, \bar{x}, y_i) \cap C_i(\bar{x}) \neq \emptyset\).

\(\text{(iii)}\) find \(\bar{x} = (\bar{x}_i)_{i \in I} \in X\) such that for each \(i \in I, \bar{x}_i \in \text{cl}S_i(\bar{x}), F_i(t_i, \bar{x}, y_i) \cap (\text{int}C_i(\bar{x})) = \emptyset\) for all \(y_i \in S_i(\bar{x})\), and for all \(t_i \in T_i(\bar{x})\).

\(\text{(iv)}\) find \(\bar{x} = (\bar{x}_i)_{i \in I} \in X\) such that for all \(i \in I, \bar{x}_i \in \text{cl}S_i(\bar{x})\) and for each \(y_i \in S_i(\bar{x})\), there exists \(t_i \in T_i(\bar{x})\).
Clearly, we have $cintA = clc(x \setminus A)$ and $X \setminus clcA = cint(x \setminus A)$. For any compact subset $K$ of $X$, we have $cintA \cap K = int_K(A \cap K)$ and $clcA \cap K = clc_K(A \cap K)$.

Definition 2.3. A set-valued mapping $T : X \to 2^Y$ is said to be transfer compactly open-valued if for $x \in X$ and for each compact subset $K$ of $Y, y \in T(x) \cap K$ implies that there exist $x' \in X$ such that $y \in int_K(T(x') \cap K)$.

**Definition 2.4.** $(Y, \{\varphi_N\})$ is said to be a FC-space if $Y$ is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \{Y\}$ where some elements in $N$ may be same, there exist a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset $D$ of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of $Y$ if for each $N = \{y_0, \ldots, y_n\} \in \{Y\}$ and for each $\{y_0, \ldots, y_n\} \subset N \cap D, \varphi_N(\Delta_k) \subset D$ where $\Delta_k = col(\{e_i : j = 0, \ldots, k\})$.

Clearly, each FC-subspace $D$ of a FC-space $(Y, \{\varphi_N\})$ is also a FC-space.

**Lemma 2.1.** Let $I$ be any index set. For each $i \in I$, let $(Y_i, \{\varphi_{N_i}\})$ be a FC-space. Let $Y = \prod_{i \in I} Y_i$ and $\varphi_N = \prod_{i \in I} \varphi_{N_i}$. Then $(Y, \{\varphi_N\})$ is also a FC-space.

**Theorem 2.1.** [13] Let $E_1, E_2$ and $Z$ be real t.v.s., $X$ and $Y$ be nonempty subset of $E_1$ and $E_2$, respectively. Let $F : X \times Y \to 2^Z, S : X \to 2^Y$ be multivalued maps.

(i) if both $S$ and $F$ are l.s.c., then $T : X \to 2^Z$ defined by $T(x) = \bigcup_{y \in S(x)} F(x, y)$ is l.s.c. on $X$;

(ii) if both $F$ and $S$ are u.s.c., with compact values, then $T$ is an u.s.c. multivalued map with compact values.

**Theorem 2.2.** [14] Let $X$ and $Y$ be topological spaces, $F : X \to 2^Y$ be a multivalued map.

(i) if $F : X \to 2^Y$ is an u.s.c. multivalued map with closed values, then $F$ is closed;

(ii) if $F$ is compact and $F : X \to 2^Y$ is an u.s.c. multivalued map with compact values, then $F(X)$ is compact.

**Proposition 2.2.** [15] Let $X$ and $Y$ be topological spaces, $F : X \to 2^Y$ be a multivalued map. Then $F$ is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ in $X$ converging to $x$, there is net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ for every $\alpha$ and $y_\alpha$ converging to $y$.

We shall use the following maximal theorem due to Ding [9].

**Theorem 2.3.** Let $I$ be an any index set. For each $i \in I$, let $(X_i, \{\varphi_{N_i}\})$ be a FC-space and let $X = \prod_{i \in I} X_i$, such that $(X, \{\varphi_N\})$ is a FC-space defined as in lemma 2.1. For each $i \in I$, let $A_i : X \to 2^Y$ such that

(i) for each $x \in X$, $A_i(x)$ is a FC-subspace of
$\exists_l$, $\exists$ for each $x \in X, x_l = \pi_i(x) \notin A_i(x)$ and $A_i^{-1} : X_l \rightarrow 2^{X_l}$ is transfer compactly open-valued.

(iii) for each $x \in X, I(x) = \{i \in I : A_i(x) \neq \emptyset\}$ is finite.

(iv) there exists a compact subset $K$ of $X$ and for each $i \in I$ and $N_i \in \langle X_i \rangle$, there exists a nonempty compact FC-subspace $L_{N_i}$ of $X_i$ containing $N_i$ such that for each $x \in X \setminus K$, there exists $y \in L_{N_i} = \prod_{i \in I} L_{N_i}$ such that for each $i \in I(x), x \in cintA_i^{-1}(\pi_i(y))$.

Then there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

3 Existance Theorems

Some existence results of a solution for the four types of system of generalized vector quasi-equilibrium problems are shown.

Theorem 3.1. Let $I$ be an any index set. For each $i \in I$, let $(X_i, \{\varphi_{N_i}\})$ and $(Y_i, \{\varphi_{M_i}\})$ be FC-spaces, let $D_i : X \rightarrow 2^{X_i}$ and $T_i : X \rightarrow 2^{Y_i}$ be set-valued maps. For each $i \in I$, assume that

(i) for each $x \in X, D_i(x)$ and $T_i(x)$ are nonempty FC-subspaces of $X_i$ and $Y_i$, respectively.

(ii) for all $(x,y) \in X \times Y$, the set $\{z_i \in X_i : F_i(x,y,z_i) \notin C_i(x)\}$ is nonempty FC-subspace of $X_i$.

(iii) for all $(x,y) \in X \times Y$ and each $x_i = \pi_i(x)$, we have $F_i(x,y,x_i) \subset C_i(x)$.

(iv) for each $i \in I$, $F_i : X \times X \times Y \rightarrow 2^{Z_i}$ is lower semicontinuous on $X \times Y$ and $C_i : X \rightarrow 2^{Z_i}$ is upper semicontinuous with closed values.

(v) for each $y_i \in X_i$ and $a_i \in Y_i$, $D_i^{-1}(y_i), T_i^{-1}(a_i)$ are compactly open.

(vi) the set $W_i = \{(x,y) \in X \times Y : x_i = \pi_i(x) \in D_i(x)$ and $y_i = \theta_i(y) \in T_i(x)\}$ is compactly closed.

(vii) for each $(x,y) \in X \times Y$, there exists $z_i \in D_i(x)$ such that $I(x,y) = \{i \in I : F_i(x,y,z_i) \notin C_i(x)\}$ is finite.

(viii) there exist nonempty and compact subsets $K \subseteq X$ and $N \subseteq Y$ and for each $i \in I$ and $B_i \subseteq \langle X_i \rangle, A_i \subseteq \langle Y_i \rangle$, there exist compact FC-subspaces $L_{B_i}$ of $(X_i)$ and $L_{A_i}$ of $(Y_i)$ containing $B_i$ and $A_i$, respectively, such that for each $(x,y) \in (X \times Y) \setminus (K \times N)$, there exists $(u,v) \in L_{B_i} \times L_{A_i}$, where $L_B = \prod_{i \in I} L_{B_i}$ and $L_A = \prod_{i \in I} L_{A_i}$, such that for each $i \in I(x,y), F_i(x,y,x_i,u) \notin C_i(x)$ and $\theta_i(u) \in T_i(x)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x}), F_i(\bar{x}_i, \bar{y}_i, z_i) \subset C_i(\bar{x})$ for all $z_i \in D_i(\bar{x})$.

Proof. For each $i \in I$, let us define a set-valued map $P_i : X \times Y \rightarrow 2^{X_i}$ by

$$P_i(x,y) = \{z_i \in X_i : F_i(x,y,z_i) \notin C_i(x)\},$$

where $\forall(x,y) \in X \times Y$. Then, $P_i(x,y)$ is a FC-subspace of $X_i$. By condition (iii), we have $x_i = \pi_i(x) \notin P_i(x,y)$. By (iv) and Theorem 2.1 it follows that for each $z_i \in x_i, P_i^{-1}(z_i)$ is compactly open. Indeed, if $(x,y) \in X \times Y \setminus P_i^{-1}(z_i)$, then there exists a net $\{x^\alpha, y^\alpha\}$ in $X \times Y \setminus P_i^{-1}(z_i)$ such that $\{x^\alpha, y^\alpha\} \rightarrow (x,y) \in X \times Y$ and $F_i(x^\alpha, y^\alpha, z_i) \subset C_i(x^\alpha)$. Let $u_i \in F_i(x_i, y_i, z_i)$, by (iv) $(x,y) \rightarrow F_i(x_i, y_i, z_i)$ is l.s.c for each $z_i \in X_i$. By Proposition 2.2, there exists a net $\{u^\alpha\}$ in $F_i(x^\alpha, y^\alpha, z_i)$ such that $u^\alpha \rightarrow u_i$. Therefore $u^\alpha \in C_i(x^\alpha)$.

Since $C_i : X \rightarrow 2^{Z_i}$ is an u.s.c multivalued map with closed values, it follows from Theorem 2.2 that $C_i$ is a closed multivalued map. Therefore, $u_i \in C_i(x)$ and $F_i(x_i, y_i, z_i) \subset C_i(x)$. We saw that $(x,y) \in X \times Y$. Therefore, $(x,y) \in X \times Y \setminus P_i^{-1}(z_i)$ and $X \times Y \setminus P_i^{-1}(z_i)$ is closed for all $z_i \in X_i$. This shows that $P_i^{-1}(z_i)$ is open for all $z_i \in X_i$. Hence, $P_i^{-1}(z_i)$ is compactly open.

By Lemma 2.1, $(X \times Y, \{\varphi_{N_i}\})$ is also a FC-space where $X \times Y = \prod_{i \in I}(X_i \times Y_i)$.

For each $i \in I$, we also define another set-valued map $S_i : X \times Y \rightarrow 2^{X_i \times Y_i}$ by

$S_i(x,y) = \{\frac{D_i(x) \times P_i(x,y) \times T_i(x), (x,y) \in W_i}{(D_i(x) \times T_i(x), (x,y) \notin W_i}\}.$

Then by (i) and $P_i(x,y)$ is a FC-subspace, for each $i \in I$ and for each $(x,y) \in X \times Y, S_i(x,y)$ is a FC-subspace of $X_i$ and so the condition (i) of Theorem 2.3 is satisfied. By (iii) and the definition of $W_i$, we have $x_i = (\pi_i(x), \theta_i(y)) \notin S_i(x,y)$ for each $i \in I$ and for any $(x,y) \in X \times Y$. For each $i \in I$ and for any $(u_i, v_i) \in X_i \times Y_i$, we have

$S_i^{-1}(u_i, v_i) = \bigcup [(D_i^{-1}(v_i) \times X) \cap [(X \times Y) \setminus W_i] \cap (D_i^{-1}(u_i) \times Y) \cap (T_i^{-1}(v_i) \times Y)]$.

By the conditions (v) and (vi), $S_i^{-1}(u_i, v_i)$ is compactly open-valued and hence $S_i^{-1}$ is transfer com-
pactly open-valued on $X_i \times Y_i$. The condition (ii) of Theorem 2.3 is satisfied. The condition (vii) implies that the condition (iii) of Theorem 2.3 holds. Note that $S_i^{-1}$ is compactly open-valued. From condition (viii), we have

$$(X \times Y) \setminus (K \times N) \subset \bigcup \{ S_i^{-1}(\pi_i(u), \theta_i(v)) : (u, v) \in L_N \times L_M \} = \bigcup \{ \text{cint} S_i^{-1}(\pi_i(u), \theta_i(v)) : (u, v) \in L_N \times L_M \}$$

and so the condition (iv) of Theorem 2.3 is satisfied.

By Theorem 2.1, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $S_i(\bar{x}, \bar{y}) = \emptyset$ for all $i \in I$. If $(\bar{x}, \bar{y}) \not\in W_i$ for some $j \not\in I$, then either $D_i(\bar{x}) = \emptyset$ or $T_i(\bar{x}) = \emptyset$ which contradicts the fact that $D_i(x)$ and $T_i(x)$ are both nonempty for each $x \in X$ and for any $i \in I$. Therefore, we have $(\bar{x}, \bar{y}) \in W_i$ for all $i \in I$, and hence for each $i \in I$, $\bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})$ and $D_i(\bar{x}) \cap T_i(\bar{x}) = \emptyset$, for all $i \in I$. Therefore, for all $i \in I$,

$$\bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x}),$$

$$F_i(\bar{x}, \bar{y}, z_i) \subset C_i(\bar{x}) \text{ for all } z_i \in D_i(\bar{x})$$

This completes the proof.

Following the same argument as Theorem 3.1, we can prove the following theorem.

**Theorem 3.2.** For each $i \in I$, assume that

(i) for each $x \in X, D_i(x), T_i(x)$ are nonempty FC-subspaces of $X_i$ and $Y_i$ respectively.

(ii) for all $(x, y) \in X \times Y$, the set $\{ z_i \in X_i : F_i(x, y, z_i) \cap C_i(x) = \emptyset \}$ is nonempty FC-subspace of $X_i$.

(iii) for all $(x, y) \in X \times Y$ and each $x_i = \pi_i(x)$ we have $F_i(x, y, x_i) \cap C_i(x) \neq \emptyset$.

(iv) for each $i \in I, F_i : X \times Y \times X_i \rightarrow 2^{X_i}$ is upper semicontinuous with compact values and $C_i : X \rightarrow 2^{X_i}$ is upper semicontinuous.

(v) for each $y_i \in X_i$ and each $a_i \in Y_i, D_i^{-1}(y_i), T_i^{-1}(a_i)$ are compactly open.

(vi) the set $W_i = \{ (x, y) \in X \times Y : x_i = \pi_i(x) \in D_i(x) \text{ and } y_i = \theta_i(y) \in T_i(x) \}$ is compact closed.

(vii) for each $(x, y) \in X \times Y$, there exists $z_i \in D_i(x)$ such that $I(x, y) = \{ i \in I : F_i(x, y, z_i) \cap C_i(x) = \emptyset \}$ is finite.

(viii) there exist nonempty and compact subsets $K \subseteq X$ and $N \subseteq Y$ and for each $i \in I$ and $B_i \subset (X_i), A_i \subset (Y_i)$, there exist compact FC-subspaces $L_{B_i}$ of $(X_i)$ and $L_{A_i}$ of $(Y_i)$ containing $B_i$ and $A_i$ respectively, such that for each $(x, y) \in X \times Y \setminus K \times N$, there exists $(u, v) \in L_B \times L_A$, where $L_B = \prod_{i \in I} L_{B_i}$ and $L_A = \prod_{i \in I} L_{A_i}$, such that for each $i \in I(x, y), F_i(x, y, \pi_i(u)) \cap C_i(x) = \emptyset$ and $\theta_i(v) \in T_i(x)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x}_i = \pi_i(\bar{x}) \in D_i(\bar{x}), \bar{y}_i = \theta_i(\bar{y}) \in T_i(\bar{x})$ and $F_i(\bar{x}, \bar{y}, z_i) \cap C_i(\bar{x}) \neq \emptyset$ for all $z_i \in D_i(\bar{x})$.

**Proof.** Let $P_i : X \times Y \rightarrow 2^{X_i}$ by $P_i(x, y) = \{ z_i \in X_i : F_i(x, y, z_i) \cap C_i(x) = \emptyset \}, \forall (x, y) \in X \times Y$.

Then, $P_i(x, y)$ is a FC-subspace of $X_i$. By condition (iii), we have $x_i = \pi_i(x) \not\in P_i(x, y)$. By (ii) and Theorem 2.1 it follows that for each $z_i \in X_i, P_i^{-1}(z_i)$ is open. Indeed, if $(x, y) \in X \times Y \setminus P_i^{-1}(z_i)$, then there exists a net $\{ x^n, y^n \} \in (X \times Y) \setminus P_i^{-1}(z_i)$ such that $\{ x^n, y^n \} \rightarrow (x, y) \in X \times Y$ and $F_i(x^n, y^n, z_i) \cap C_i(x^n) \neq \emptyset$. Let $u^n \in F_i(x^n, y^n, z_i) \cap C_i(x^n)$. By (iv) and Theorem 2.2 that for each $z_i \in X_i, (x, y) \rightarrow F_i(x, y, z_i)$ is an u.s.c multivalued map with compact values. It suffices to find a subset $\{ u_i^n \}$ of $u^n$, which converges to some $u_i \in F_i(x, y, z_i)$. Since for each $z_i \in X_i$, the multivalued map $(x, y) \rightarrow F_i(x, y, z_i)$ is u.s.c with compact values, it follows from Theorem 2.2 that for each fixed $z_i \in X_i, (x, y) \rightarrow F_i(x, y, z_i)$ and $C_i$ are closed. Therefore, $(x, y) \in X \times Y$ and $u_i \in F_i(x, y, z_i) \cap C_i(x) \neq \emptyset$. This shows that $X \setminus P_i^{-1}(z_i)$ is closed for each $z_i \in X_i$. Hence $P_i^{-1}(z_i)$ is open for each $z_i \in X_i$.

By lemma 2.1, $(X \times Y, (\{ \varphi_Y \})_i)$ is also a FC-space where $X \times Y = \coprod_{i \in I}(X_i \times Y_i)$.

For each $i \in I$, we also define another set-valued map $S_i : X \times Y \rightarrow 2^{X_i}$ by

$$S_i(x, y) = \begin{cases} \{ D_i(x) \times P_i(x, y) \times T_i(x) \} \subset X_i, & (x, y) \in W_i; \\ D_i(x) \times T_i(x), & (x, y) \not\in W_i; \end{cases}$$

Then by (i) and $P_i(x, y)$ is a FC-subspace, for each $i \in I$ and for each $(x, y) \in X \times Y, S_i(x, y)$ is a FC-subspace of $X_i$ and so the condition (i) of Theorem 2.3 is satisfied. By (b) and the definition of $W_i$, we have $(x_i, y_i) = (\pi_i(x), \theta_i(y)) \not\in S_i(x, y)$ for each $i \in I$ and for any $(x, y) \in X \times Y$. For each $i \in I$ and for any $(u_i, v_i) \in X_i \times Y_i$, we have

$$S_i^{-1}(u_i, v_i) = \{ P_i^{-1}(u_i) \cap D_i^{-1}(u_i) \times Y \} \cap T_i^{-1}(v_i) \cup \left( (X \times Y) \setminus W_i \right) \cap (D_i^{-1}(u_i) \times Y) \cap T_i^{-1}(v_i) \times X \}$$
By the conditions (v) and (vi), \( S_i^{-1}(u, v) \) is compactly open-valued and hence \( S_i^{-1} \) is transfer compactly open-valued on \( X_i \times Y_i \). The condition (ii) of Theorem 2.3 is satisfied. The condition (viii) implies that the condition (iii) of Theorem 2.3 holds. Note that \( S_i^{-1} \) is compactly open-valued. From condition (viii), we have

\[
(X \times Y) \setminus (K \times N) \subset \bigcup \{ S_i^{-1}(\pi_i(u), \theta_i(v)) : (u, v) \in L_N \times L_M \}
\]

\[
= \bigcup \{ \text{cnt} S_i^{-1}(\pi_i(u), \theta_i(v)) : (u, v) \in L_N \times L_M \}
\]

and so the condition (iv) of Theorem 2.3 is satisfied.

By Theorem 2.3, there exists \((\hat{x}, \hat{y}) \in X \times Y\) such that \( S_i(\hat{x}, \hat{y}) = \emptyset \) for all \( i \in I \). If \((\hat{x}, \hat{y}) \notin W_i \) for some \( j \notin I \), then either \( D_j(\hat{x}) = \emptyset \) or \( T_i(\hat{x}) = \emptyset \).

Then there exists \((\hat{x}, \hat{y}) \in X \times Y\) such that for each \( i \in I \), \( \hat{x}_i = \pi_i(\hat{x}) \in D_i(\hat{x}), \hat{y}_i = \theta_i(\hat{y}) \in T_i(\hat{x}), F_i(\hat{x}, \hat{y}, z_i) \cap (-\text{int}C_i(x)) = \emptyset \) for all \( z_i \in D_i(\hat{x}) \).

Then there exists \((\hat{x}, \hat{y}) \in X \times Y\) such that for each \( i \in I \), \( \hat{x}_i = \pi_i(\hat{x}) \in D_i(\hat{x}), \hat{y}_i = \theta_i(\hat{y}) \in T_i(\hat{x}), F_i(\hat{x}, \hat{y}, z_i) \cap (-\text{int}C_i(x)) = \emptyset \) for all \( z_i \in D_i(\hat{x}) \).

Theorem 3.4. For each \( i \in I \), assume that

(i) for each \( x \in X, D_i(x), T_i(x) \) are nonempty compact FC-subspaces of \( X_i \) and \( Y_i \), respectively.

(ii) for all \((x, y) \in X \times Y\), the set \( \{z_i \in X_i : F_i(x, y, z_i) \cap (-\text{int}C_i(x)) = \emptyset\} \) is nonempty FC-subspace of \( X_i \).

(iii) for all \((x, y) \in X \times Y\), each \( x_i = \pi_i(x) \) we have \( F_i(x, y, x_i) \cap (-\text{int}C_i(x)) = \emptyset \).

(iv) for each \( i \in I \), \( F_i : X \times Y \times X_i \twoheadrightarrow 2^{Z_i} \) is upper semicontinuous on \( X \times Y \) and \( C_i : X \twoheadrightarrow 2^{Z_i} \), is upper semicontinuous with closed values.

(v) for each \( y_i \in X_i \) and each \( a_i \in Y_i \), \( D_i^{-1}(y_i), T_i^{-1}(a_i) \) are compactly open.

(vi) the set \( W_i = \{(x, y) \in X \times Y : x_i = \pi_i(x) \text{ and } y_i = \theta_i(y) \in T_i(x)\} \) is compactly closed.

(vii) for each \( (x, y) \in X \times Y \), there exists \( z_i \in D_i(x) \) such that \( I(x, y) = \{i \in I : F_i(x, y, z_i) \cap (-\text{int}C_i(x)) \neq \emptyset\} \) is finite.

(viii) there exist nonempty and compact subsets \( K \subseteq X \) and \( N \subseteq Y \) and for each \( i \in I \) and \( B_i \subset \langle X_i \rangle, A_i \subset \langle Y_i \rangle \), there exist compact FC-subspaces \( L_{B_i} \) of \( \langle X_i \rangle \) and \( L_{A_i} \) of \( \langle Y_i \rangle \) containing \( B_i \) and \( A_i \), respectively, such that for each \((x, y) \in (X \times Y) \setminus (K \times N)\), there exists \((u, v) \in L_B \times L_A \), where \( L_B = \prod_{i \in I} L_{B_i} \) and \( L_A = \prod_{i \in I} L_{A_i} \), such that for each \( i \in I(x, y), F_i(x, y, \pi_i(u)) \cap (-\text{int}C_i(x)) \neq \emptyset \) and \( \theta_i(v) \in T_i(x) \).

Then there exists \((\hat{x}, \hat{y}) \in X \times Y\) such that for each \( i \in I \), \( \hat{x}_i = \pi_i(\hat{x}) \in D_i(\hat{x}), \hat{y}_i = \theta_i(\hat{y}) \in T_i(\hat{x}), F_i(\hat{x}, \hat{y}, z_i) \cap (-\text{int}C_i(x)) = \emptyset \) for all \( z_i \in D_i(\hat{x}) \).
References


First Author received the BS degree in mathematics from Sichuan Normal University in 2004. He received the MS degree in applied mathematics from the Tianjin Polytechnic University in 2007. Now, he is also a mathematical teacher of Tianjin Agricultural University. His current interests are in the area of pattern recognition, optimization method and machine learning. He has published several technical papers in these areas.

Second Author received the BS degree in mathematics from Hebei Normal University in 2003. She received the MS degree in applied mathematics from the Tianjin Polytechnic University in 2006. Now, she is also a mathematical teacher of Tianjin University of Technology. Her current interests are in the area of pattern recognition, optimization method and machine learning. She has published several technical papers in these areas.