# Numerical simulation of spatio-temporal model: case of SIR epidemic model

**Imane El Berrai, Jamal Bouyaghroumni and Abdelouahed Namir**

**LAMS (Laboratory of analysis, modeling and simulation),**
**Department of mathematics and computer sciences, Faculty of sciences**
**Ben M'Sik-Casablanca, Hassan II University Mohammedia-Casablanca Morocco.**

## Abstract

In this work we study a model to describe epidemics, the SIR (Susceptible- Infectious- Recovered) model. We improve the SIR model in order to take into account spatial behavior of the infected and susceptible populations by adding a diffusion term to the equations. The SIR epidemic model is considered in bidimensional case. To solve the considered system we use the finite difference approach based on explicit Euler schema. In addition, we present the numerical results.

*Keywords:* *SIR model, Spread of epidemics, Numerical simulation, dissemination.*

## 1. Introduction

In 1927, W. Kermack and A. McKendrick [14]first proposed a model to explain the rapid rise and fall in the number of infected patients observed in epidemics such as the plague (London 1665-1666, Bombay 1906) and cholera (London 1865)" [8]. In 1979, M. Anderson and M. May resurrected the Kermack, McKendrick model, as known as the SIR model, for the spread of infectious diseases More recently, this simple epidemiological model consisting of three coupled ordinary differential equations has been used for a variety of epidemics, including HIV and SARS). Due to its applicability, it has become the foundation for more complex, more realistic epidemiological models.

A detailed history of mathematical epidemiology and basics of SIR epidemic models may be found in the classical books of Bailey [2], Murray [19], and Anderson and May [1]. After Kermack- McKendrick model, different epidemic models have been proposed and studied in the literature (see Capasso and Serio [6], Hethcote and Tudor [13], Liu et al. [15][16], Hethcote et al. [12], Hethcote and van den Driessche

[9], Derrick and van den Driessche [7], Beretta and Takeuchi [5][4], Beretta et al. [3], Ma et al. [17][18], Ruan and Wang [21], Song and Ma [22], Song et al. [23], D'Onofrio et al. [20], Xiao and Ruan [24].

The population is divided into disjoint classes which change with time. The susceptible class consists of those individuals who can incur the disease but are not yet infective. The infective class consists of those who are removed from the susceptible - infective interaction by recovery with immunity, isolation, or death. The fractions of the total population in these classes are denoted by S(t), I(t) and R(t) respectively.

In the epidemiological models here, the following assumptions are made:
1) The only way a person can leave the susceptible group is to become infected.
2) The only way a person can leave the infected group is to recover from the disease.
3) Once a person has recovered, the person received immunity.
4) Age, sex, social status, and race do not affect the probability of being infected.

The remaining parts of this paper are organized as follows: section 2 presents the epidemic model. In section 3 we study the behavior of the epidemic in each location of a territory, by adding a Laplacian which corresponds to the spatial dispersion of individuals susceptible (infected respectively), it being modeled by a diffusion phenomenon is to say a result of movements of individuals susceptible (infected respectively). We present some efficient algorithms for solving the discretized problem on Matlab In section 4. The five section to put spotlight on numerical results. The last section provides concluding remarks.

## 2. The Epidemic model

The model to be studied takes the following form:

$$\begin{cases} \dot{S}(t) = r_c S(t)\left(1 - \dfrac{S(t)}{k}\right) - \dfrac{\alpha S(t)I(t)}{1 + aI(t)} \\ \dot{I}(t) = \dfrac{\alpha S(t)I(t)}{1 + aI(t)} - \gamma I(t) \end{cases} \quad (1)$$

The model has a susceptible group designated by S, an infected group I, and a recovered group R with permanent immunity, $r_c$ is the intrinsic growth rate of susceptible, k is the carrying capacity of the susceptible in the absence of infective, α is the maximum values of per capita reduction rate of S du to I, a is half saturation constants, γ is the natural recover rate from infection.

## 3. Spatio-temporal model

In recent years, the problems of infection are increasingly present in our daily lives. Each individual is required to meet a number of people in a day either in the workplace or at various outlets (travel, cinema, shopping centers ...). We can cite models of spatial diffusion for Hunter [10], Gilg [11] and El Berrai et al [8] which deal with different types of epidemic. To model the spread of the epidemic for this SIR model it is assumed here that the population moves: if the population is distributed in several cities, individuals can become infected within the same city, but more likely an individual can move in a city where it is an infection, then bring the disease in his hometown, or another infected individual can move in another city to spread the disease ...
It will be modeled by the following system:

$$\begin{cases} \dot{S}(t) = \nabla^2 S + r_c S(t)\left(1 - \dfrac{S(t)}{k}\right) - \dfrac{\alpha S(t)I(t)}{1 + aI(t)} \\ \dot{I}(t) = \nabla^2 I + \dfrac{\alpha S(t)I(t)}{1 + aI(t)} - \gamma I(t) \end{cases} \quad (2)$$

The Laplacian is the spatial dispersion of individuals susceptible (infected respectively), the latter being modeled by a dissemination phenomenon that is to say, a result of movements of individuals susceptible (infected respectively).

In the stationary problem, the system does not depend on time then we obtain:

$$\begin{cases} \dfrac{\partial^2 S}{\partial x^2} + \dfrac{\partial^2 S}{\partial y^2} = -r_c s\left(1 - \dfrac{s}{k}\right) + \dfrac{\alpha si}{1 + ai} \\ \dfrac{\partial^2 I}{\partial x^2} + \dfrac{\partial^2 I}{\partial y^2} = -\dfrac{\alpha si}{1 + ai} + \gamma I \end{cases} \quad (3)$$

When the system depends on the time we obtained:

$$\begin{cases} \dot{S}(t) = \dfrac{\partial^2 S}{\partial x^2} + \dfrac{\partial^2 S}{\partial y^2} + r_c s\left(1 - \dfrac{s}{k}\right) - \dfrac{\alpha si}{1 + ai} \\ \dot{I}(t) = \dfrac{\partial^2 I}{\partial x^2} + \dfrac{\partial^2 I}{\partial y^2} + \dfrac{\alpha si}{1 + ai} - \gamma I \end{cases} \quad (4)$$

## 4. Approximation of the system

In this part, we try to observe numerically the solutions of the system in the form of travelling waves to model an epidemic. For that, we will approximate the solutions.

We study the spread of an epidemic in dimension 2, that is to say people moves on a plane (digitally, it is restricted to the polygon ( ] $x_{min}$, $x_{max}$ [ ] $y_{min}$, $y_{max}$ [).

we consider the problem to the limits associated with the system:

$$\begin{cases} \dot{S}(t) = \dfrac{\partial^2 S}{\partial x^2} + \dfrac{\partial^2 S}{\partial y^2} + r_c s\left(1 - \dfrac{s}{k}\right) - \dfrac{\alpha si}{1 + ai} \\ \dot{I}(t) = \dfrac{\partial^2 I}{\partial x^2} + \dfrac{\partial^2 I}{\partial y^2} + \dfrac{\alpha si}{1 + ai} - \gamma I \end{cases}$$

We fix $M_x > 0$ and $M_y > 0$ the number of domestic points in polygon ] $x_{min}$, $x_{max}$ [ ] $y_{min}$, $y_{max}$ [. We note the discretization in space $\Delta_x = h_x = \frac{x_{max} - x_{min}}{M+1}$ and $\Delta_y = h_y = \frac{y_{max} - y_{min}}{M+1}$ with $x_i = x_{min} + i\, h_x$ and $y_i = y + i\, h_y$. We divide the interval [0; T] into N sub-intervals, defined by: $0 = t_0 < t_1 < t_2 < \dots < t_n = $ T with $l = \frac{T}{N}$. We will calculate an approximation of the solution, denoted by $\left(S_{i,j}^n, I_{i,j}^n\right)$.

### 4.1. Proposition

We have a space-time discretization:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}$$

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1}^n - u_{i,j}^n}{\Delta t}$$

IJCSI International Journal of Computer Science Issues, Vol. 11, Issue 2, No 1, March 2014
ISSN (Print): 1694-0814 | ISSN (Online): 1694-0784
www.IJCSI.org

107

We put:

$$H(S, I) = \frac{S_{i+1,j}^n + S_{i-1,j}^n}{\Delta x^2} + \frac{S_{i,j+1}^n + S_{i,j-1}^n}{\Delta y^2} + S_{i,j}^n [\, r_c \left(1 - \frac{S_{i,j}^n}{k}\right) - \frac{\alpha I_{i,j}^n}{1+a I_{i,j}^n}$$
$$-2(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2})\,].$$

$$G(S, I) = \frac{I_{i+1,j}^n + I_{i-1,j}^n}{\Delta x^2} + \frac{I_{i,j+1}^n + I_{i,j-1}^n}{\Delta y^2} + I_{i,j}^n[\frac{\alpha I_{i,j}^n}{1+a I_{i,j}^n} - \gamma - 2(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2})\,].$$

So we obtain :

$$\begin{cases} S_{i,j}^{n+1} &= \Delta t\, H(S, I) + S_{i,j}^n \\ I_{i,j}^{n+1} &= \Delta t\, G(S, I) + I_{i,j}^n \end{cases} \tag{5}$$
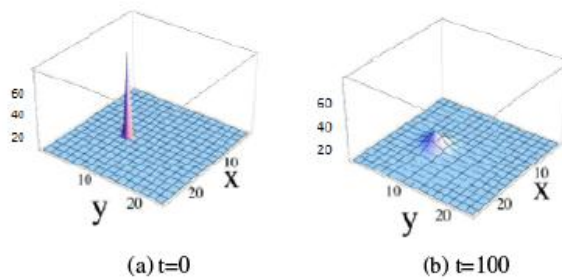
## 5. Numerical simulation

For the numerical simulations we use the following data.
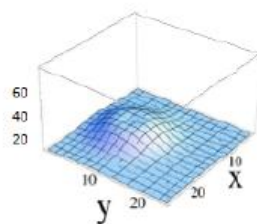
Table 1: Value of the parameters

| Parameter | $S_0$ | $I_0$ | a | α | γ | k | $r_c$ |
|-----------|-------|-------|-----|------|-------|-----|-------|
| Value | 50 | 30 | 2.3 | 1.49 | 0.611 | 100 | 2.5 |

We show snapshots of the spatial and temporal evolution of the infected populations for several times.
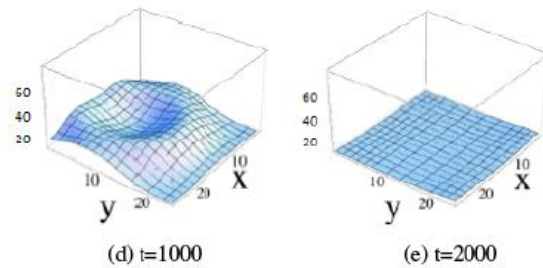


(a) t=0

(b) t=100



(c) t=550



(d) t=1000

(e) t=2000

Figure 1: Snapshots of the spatial and temporal evolution of the infected populations.

Initially at time t = 0, we introduce a proportion of infected. At t = 100 in the figure below the infected population begins to disseminate. At t = 550 the dissemination is still amplifies. At t = 1000 there is no longer a sick in the center. Finally, at t = 2000, the epidemic is finite. The infected population is constant equal to 0.
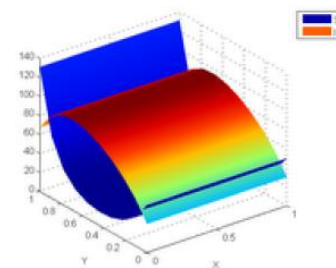
In the stationary case we find the following figure:



Figure 2: Stationary Solutions

## 6. Conclusions

The Spatiotemporal Epidemiological Model (STEM) tool is designed to help scientists and public health officials create and use spatial and temporal models of emerging infectious diseases. These models can aid in understanding and potentially preventing the spread of such diseases. Its disease model computations are based on compartment models that assume an individual is in a particular state,

either susceptible (S), infectious (I), or recovered (R), in classic SI(S), SIR(S), disease models. In this work, the SIR epidemic model of infectious diseases in populations is considered in bidimensional case. This model permits one hand to modeled the dynamic of spread of the epidemic, and the other hand we was permits to generalize the study proposed by El Berrai et al[8]. Besides, we approximate the considered model using the Euler schema and we present the obtained numerical results for differents values of time.

## References

[1] R.M. Anderson, R.M. May, Infectious Diseases of Humans: Dynamics and Control, Oxford University Press, Oxford, 1998.

[2] N.T.J. Bailey, The Mathematical Theory of Infectious Diseases, Griffin, London, 1975.

[3] E. Beretta, T. Hara, W. Ma, Y. Takeuchi, Global asymptotic stability of a SIR epidemic model with distributed time delay, Nonlinear Anal., 47, pp. 4107- 4115, 2001.

[4] E. Beretta, Y. Takeuchi, Convergence results in SIR epidemic model with varying population sizes, Nonlinear Anal., 28, pp. 1909-1921, 1997.

[5] E. Beretta, Y. Takeuchi, Global stability of a SIR epidemic model with time delay, J. Math. Biol., 33, pp. 250-260, 1995.

[6] V. Capasso, G. Serio, A generalization of the Kermack-McKendrick deterministic epidemic model, Math. Biosci., 42, pp. 43-61, 1978.

[7] W.R. Derrick, P. van den Driessche, A disease transmission model in nonconstant population, J. Math. Biol., 31, pp. 495-512, 1993.

[8] I. EL Berrai, J. Bouyaghroumni, A. Namir, Dissemination of Epidemic for SIR Model, Applied Mathematical Sciences, Vol. 7, 2013, no. 136, 6793 - 6800.

[9] H.W. Hethcote, P. van den Driessche, Some epidemiological model with nonlinear incidence, J. Math. Biol., 29, pp. 271-287, 1991.

[10] HUNTER J.M. et YOUNG J.C., Diffusion of influenza in England and Wales, in annals of the Association of American Geographers, 61, p. 637-653, 1971.

[11] GILG A.W., A study in agricultural disease diffusion : the case of the 1970-71 fowl-pest epidemic, in Institute of British Geographers, Publications, 59, p. 77-97, 1973. 1996.

[12] H.W. Hethcote, M.A. Lewis, P. van den Driessche, An epidemiological model with delay and a nonlinear incidence rate, J. Math. Biol., 27, pp. 49-64, 1989.

[13] H.W. Hethcote, D.W. Tudor, Integral equation models for endemic infectious diseases, J. Math. Biol., 9, pp. 37-47, 1980.

[14] Kermack, W. O. and McKendrick, A. G. (1927) Contributions to the mathematical theory of epidemics, part i. Proceedings of the Royal Society of Edinburgh. Section A. Mathematics . 115 700-721.

[15] W.M. Liu, H.W. Hethcote, S.A. Levin, Dynamical behaviour of epidemiological models with nonlinear incidence rates, J. Math. Biol., 25, pp. 359-380, 1987.

[16] W. M. Liu, S. A. Levin, Y. Iwasa,Influence of nonlinear incidence rates upon the behaviour of SIRS epidemiological models, J. Math. Biol., 23, pp. 187- 204, 1986.

[17] W. Ma, M. Song, Y. Takeuchi, Global stability of an SIR epidemic model with time-delay, Appl. Math. Lett., 17, pp. 1141-1145, 2004.

[18] W. Ma, Y. Takeuchi, T. Hara, E. Beretta, Permanence of an SIR epidemic model with distributed time delays, Tohoku Math. J., 54, pp. 581-591, 2002.

[19] J.D. Murray, Mathematical Biology, Springer-Verlag, New York, 1993.

[20] A. D'Onofrio, P. Manfredi, E. Salinelli, Bifurcation thresholds in an SIR model with information-dependent vaccination, Mathematical Modelling of Natural Phenomena, 2, pp. 23-38, 2007.

[21] S. Ruan,W.Wang, Dynamical behaviour of an epidemic model with nonlinear incidence rate, J. Differ. Equations, 188, pp. 135-163, 2003.

[22] M. Song,W. Ma, Asymptotic properties of a revised SIR epidemic model with density dependent birth rate and tie delay, Dynamic of Continuous, Discrete and Impulsive Systems, 13, pp. 199-208, 2006.

[23] M. Song, W. Ma, Y. Takeuchi, Permanence of a delayed SIR epidemic model with density dependent birth rate, J. Comput. Appl. Math., 201, pp. 389-394, 2007.

[24] D. Xiao, S. Ruan, Global analysis of an epidemic model with nonmonotone incidence rate, Math. Biosci., 208, pp. 419-429, 2007.