New Generalized Interval Arithmetic and its applications to structural mechanics and electrical circuits.

El-Owny, Hassan Badry Mohamed A.

CIT College, Taif University, 21974 Taif, KSA.

Mathematics department, Faculty of Science,
Aswan University, 81528 Aswan, Egypt.

Abstract

In the paper we solve parametric linear systems of equations whose coefficients are , in the general case, nonlinear functions of interval parameters. Here solution means that we enclose the set of all solutions, the so-called parametric solution set, obtained when all parameters are allowed to vary within their intervals. This task appears in many scientific and engineering problems involving uncertainties. A C-XSC implementation of a parametric fixed-point iteration method for computing an outer enclosure for the solution set is proposed in this paper .This method requires to bound the range of a multivariate function over a given box and often delivers intervals which are too wide for practical applications. We computed tight enclosures of the parametric solution set by using a new generalized interval arithmetic which is an arithmetic for intervals (which are representing uncertainties). The most important property of this method is to reduce the effect of the dependency problem which is inherent in the computation with standard interval arithmetic. We used the new arithmetic to tightly bound the range of a multivariate nonlinear function over a box, a task to which many problems in mathematics and its applications can be reduced. We applied the new bounding technique to improve the efficiency of the solution for parametric systems. Numerical examples illustrating the applicability of the proposed method are solved, and compared with other methods.

Keywords: *Interval Arithmetic, dependency problem, validated interval software, parametric interval systems, C-XSC, electrical circuits, structural mechanics*.

1. Introduction

In many engineering design problems, linear prediction problems, and models in operation research, etc. there are often complicated dependencies between the coefficients. Engineering problems that involve such parametric linear systems may stem from structural mechanics, the design of electrical circuits, resistive networks, and robust Monte Carlo simulation, etc.. The source of parametric uncertainty is often the lack of precise data which may result from a lack of knowledge or an inherent variability in the parameters. Many sources of uncertainty exist in models for the analysis of structural mechanics problems and electrical circuit. These include, e.g., measurement imprecision, manufacturing imperfections, and round-off errors. An uncertain quantity is often assumed to be unknown but bounded, i.e., lower and upper bounds for this quantity can be provided (without assigning any probability distribution). Therefore, these quantities can be represented by intervals. Interval arithmetic provides the means to keep track of such uncertainties throughout the whole computation. Consequently, the result, which is again an interval quantity, is guaranteed to contain the exact result.

Scientific and engineering problems described by systems of linear equations involving uncertain model parameters include problems in engineering analysis or design [1,2,3,4,5,6,7], control engineering [8], the design of electrical circuits [9], resistive networks, and robust Monte Carlo simulation [10,11], etc. Significant research in this field is directed towards the use of intervals [12, 13] to represent the uncertain quantities in such systems.

The solution of a parametric linear system, the so-called parametric solution set (abbreviated by $\sum^p$ henceforth) is the set of all solutions which are obtained when all parameters are allowed to vary within their intervals. It can be described explicitly only in very simple cases. Therefore, one attempts to find the smallest axis-aligned box in $\mathbb{R}^n$ containing $\sum^p$. Since even this set can only be found easily in some special cases, it is more practical to compute a tight outer approximation to this box.

The parametric residual iteration is a self-verified method for bounding the parametric solution set. This is a general-purpose method since it does not assume any particular structure among the parameter dependencies. The method originates in the inclusion theory for nonparametric problems, which is discussed in many works. The basic idea of combining the Krawczyk operator [14] and the existence test by Moore [13] is further elaborated by S. Rump who proposes several improvements leading to inclusion theorems for the interval solution. S. Rump gives in [15] a straightforward generalization to linear systems with affine-linear dependencies in the matrix and the right hand side. With obvious modifications, the results can also be applied directly to linear systems involving nonlinear dependencies between the parameters.

Consider a parametric system,

$$A(p) \cdot x = b(p), \quad (1)$$

where $A(p) \in \mathbb{R}^{n\times n}$, and $b(p) \in \mathbb{R}^n$ depend on a parameter vector $p \in \mathbb{R}^n$. The elements of $A(p)$ and $b(p)$ are in general, nonlinear function of k parameters $p_1, \cdots, p_k$

$$a_{ij}(p) = a_{ij}(p_1, \cdots, p_k)$$
$$b_i(p) = b_i(p_1, \cdots, p_k), \quad (i, j = 1,2, \cdots, n) \quad (2)$$

The parameters are considered to be unknown or uncertain and varying within prescribed intervals $p \in [p] = (p_1, \cdots, p_k)^\intercal$.

When p varies within a range $[p] \in \mathbb{IR}^k$, the set of all solution to all $A(p) \cdot x = b(p)$, $p \in [p]$, is called parametric solution set, and is denoted by

$$\textstyle\sum^p := \sum(A(p), b(p), [p]) :=$$
$$\{x \in \mathbb{R}^n | A(p) \cdot x = b(p) \text{ for some } p \in [p]\} \quad (3)$$

Since the solution set has a complicated structure (it does not even need to be convex), which is difficult to find., one looks for the interval hull $\lozenge(\sum)$ here $\sum$ is a nonempty bounded subset of $\mathbb{R}^n$. For $\sum \subseteq \mathbb{R}^n$, define $\lozenge : P\mathbb{R}^n \longrightarrow \mathbb{IR}^n$ by[1]

$$\lozenge(\textstyle\sum) := [\inf \sum, \sup \sum] = \bigcap\{[x] \in \mathbb{IR}^n | \sum \subseteq [x]\}.$$

The calculation of $\lozenge(\sum)$ is also quite expensive.

Since it is quite expensive to obtain $\sum$ or $\lozenge(\sum)$, it would be a more realistic task to find an interval vector $[y] \in \mathbb{IR}^n$ which tightly encloses $[y] \supseteq \lozenge(\sum^p) \supseteq \sum^p$.

Probably the first general purpose method for computing outer (and inner) bounds for the interval hull of $(\sum^p)$ is based on the fixed-point interval iteration theory developed by S. Rump. In [15] he applies the general verification theory for system of nonlinear equations to the solution of parametric linear systems involving affine-linear dependencies. This method was generalized to nonlinear parameter dependencies; however, it is required to find tight bounds on the ranges of nonlinear functions. Meanwhile, there were many attempts to construct suitable methods for solving parameter dependent interval linear systems [16, 17]. Kolev proposed a directed method [18, 19] and an iterative method [16] for computing an enclosure of the solution set. Parameterized Gauss-Seidel iteration was employed by Popova [20]. A direct method was given by Skalna in [21], and a monotonicity approach in [22]. Inner and outer approximations by a fixed-point method was developed and implemented in [23]. A *Mathematica* package for solving parametric interval systems is introduced in [24]. We do not intend to give here a complete overview of these methods. Most of the methods developed so far address linear systems involving affine-linear dependencies between the parameters and only few articles study the general case of nonlinear parameter dependency [25, 5].

The goal of this work is to introduce a new C-XSC software (C- for Extended Scientific Computing) [26] for a new generalized interval arithmetic to tightly enclose multivariate nonlinear functions, and use it to find the solution set of parametric interval systems, i.e., interval vectors, which contain all possible solutions of this system. We will compare our method to other methods.

In Section 2 the dependency problem is introduced. The rest of the paper is organized as follows. The new generalized form and its arithmetics is introduced in Section 3. The theoretical framework of the paper is presented in Section 4. The main results of this paper is presented in Section 5. Numerical experiments illustrating the features of the proposed method are provided in Section 6. The paper ends with concluding remarks.

We use the following notations $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times n}$, $\mathbb{IR}, \mathbb{IR}^n, \mathbb{IR}^{n\times n}$, to denote the set of real numbers, the set of real vectors with *n* components, the set of real *n*×*n* matrices, the set of intervals, the set of interval vectors with *n* components and the set of *n*×*n* interval matrices, respectively. For a real interval $[x]$ we mean a compact interval $[x] := [\underline{x}, \overline{x}] := \{x \in \mathbb{R} | \ \underline{x} \le x \le \overline{x}\}$, Where $\underline{x}$ and $\overline{x}$ denote the lower and upper bounds of the interval $[x]$, respectively.

For a real interval $[x]$ define the mid-point

$$\hat{x} = \text{mid}([x]) := {(\overline{x} + \underline{x})}/{2}$$

and the radius

$$\text{rad}([x]) := {(\overline{x} - \underline{x})}/{2} ,$$

Definition of real intervals and operation with such intervals can be found in a number of references [12,13]. However, we present the main interval arithmetic operation. For Definition of real intervals and operation with such intervals can be found in a number of references [12,13]. However , we present the main interval arithmetic operations. For $[x]$and $[y] \in \mathbb{IR}^n$

$$[x] + [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$
$$[x] - [y] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$$
$$[x] \cdot [y] = [\min(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}), \max(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y})] \quad (4)$$
$$1/[y] = \left[1/\overline{y}, 1/\underline{y}\right] \quad \text{if } 0 \notin [y]$$
$$[x]/[y] = [x] \cdot \left[1/\overline{y}, 1/\underline{y}\right] \quad \text{if } 0 \notin [y]$$

2. Dependency problem

The dependency problem arises when one or several variables occur more than once in an interval expression. Dependency may lead to catastrophic overestimation in interval computations. For example, if the interval $[x] = [1,2]$ is subtracted from itself $[x] - [x] = [-1,1]$ is obtained as the result and not the interval $[0,0]$ as expected. Actually, interval arithmetic cannot

[1] $P\mathbb{R}^n$ is the power set over $\mathbb{R}^n$. Given a set S, the power set of S is the set off all subset of S

recognize the multiple occurrence of the same variable *x*. The result is $\{x - y|\, x \in [x],\ y \in [y]\}$ instead of $\{x - x|\, x \in [x]\}$. In general, when a given variable occurs more than once in an interval computation, it is treated as a different variable in each occurrence.

For a less extreme example, take $f(x) = (10 + x) \cdot (10 - x)$ for $x \in [x] = [-1, 1]$ Using the basic formulas (4) , we get

$$10 + [x] = [9,11]$$
$$10 - [x] = [9,11]$$
$$(10 + [x]) \cdot (10 - [x]) = [81,121].$$

The interval formulas give an interval whose diameter is 40, whereas the exact interval result $f([x]) = [99, 100]$ has a diameter of only 1. Note that when one operand in the product $(10 + x) \cdot (10 - x)$ is at the maximum value 11, the other must be at the minimum value 9; the combination $9 \cdot 9$ and $11 \cdot 11$, which gave the extreme values of $F([x])$, never occur.

A simple remedy for this example is to rewrite $(10 + x) \cdot (10 - x) = 100 - x^2$, which has only one occurrence of the variable *x*. An interval computation of this new expression will give the exact result. Unfortunately, this remedy is often impossible to apply in practice.

Several other methods have been proposed to attack the dependency problem. The main class of methods is known as centered forms [9], in several incarnations and generalizations, such as mean-value form [9] and slopes [14].

The purpose of a new generalized interval arithmetic method is to reduce the effect of the dependency problem when computing with standard interval arithmetic, and in addition, it provides the enclosure of the range of a nonlinear interval function by linear interval forms.

3. New Generalized Interval Arithmetic

For our purposes, it is convenient to represent an interval $[x] = [\underline{x}, \overline{x}]$ in the form $[x] = c + [u]$ where *c* is the mid-point of [*x*], $[u] = [-r, r]$ is symmetrical interval, and *r* is the radius of [*x*]. Thus an arbitrary point $x \in [x]$ may be expressed as $x = c + \zeta$ where $\zeta \in [-r, r]$ and $r \geq 0$.

Definition 1. [9,27]: *A generalized interval* $[\breve{x}]$ *is given by*

$$[\breve{x}] = [m^x] + \textstyle\sum_{i=1}^n \zeta_i [u_i^x] \qquad (5)$$

where $[m^x]$ *and* $[u_i^x]$, $(i = 1, 2, \cdots, n)$ *are computed intervals and* $\zeta_i \in [-r_i, r_i]$.

When we reduce the generalized interval in (5) to an ordinary interval, we obtain

$$\mathrm{reduce}([\breve{x}]) = \mathrm{reduce}([m^x] + \sum_{i=1}^n [-r_i, r_i][u_i^x])$$
$$:= [m^x] + [-1,1] \sum_{i=1}^n r_i u_i^x$$

where $u_i^x := |[u_i^x]|$. In general, the absolute value of an interval $[x] = [\underline{x}, \overline{x}]$ is defined as $|[x]| := \max(|\underline{x}|, |\overline{x}|)$. Conversely, any ordinary interval can be represented by a generalized interval. The ordinary interval $[x] = [\underline{x}, \overline{x}]$ can be represented as the generalized interval $[\breve{x}] = [m^x] + \zeta_1 [u_1^x]$, where $[m^x] := \left[\frac{\underline{x}+\overline{x}}{2}, \frac{\underline{x}+\overline{x}}{2}\right]$, $\zeta_1 \in \left[\frac{\overline{x}-\underline{x}}{2}, \frac{\overline{x}-\underline{x}}{2}\right]$ and $[u_1^x] := [1,1]$. In general, if we have an interval vector $[x] := ([x_1], \cdots, [x_n])^\intercal \in \mathbb{IR}^n$, the *j*-th interval can be represented with the generalized interval form

$$[\breve{x}_j] = [m^{x_j}] + [0,0]\zeta_1 + \cdots + [0,0]\zeta_{j-1} + [1,1]\zeta_j + [0,0]\zeta_{j+1} + \cdots + [0,0]\zeta_n$$
$$= [m^{x_j}] + [1,1]\zeta_j$$

Assume two generalized intervals $[\breve{x}]$ and $[\breve{y}]$ are expressed as $[\breve{x}] = [m^x] + \sum_{i=1}^n \zeta_i [u_i^x]$ and $[\breve{y}] = [m^y] + \sum_{i=1}^n \zeta_i [u_i^y]$ respectively. We now consider the arithmetic operations applied to these generalized intervals.

Addition and Subtraction

The sum (difference) of $[\breve{x}]$ and $[\breve{y}]$ is another generalized interval $[\breve{z}] = [m^z] + \sum_{i=1}^n \zeta_i [u_i^z]$.

It holds

$$[\breve{x}] \pm [\breve{y}] = [m^x] \pm [m^y] + \textstyle\sum_{i=1}^n \zeta_i ([u_i^x] \pm [u_i^y]) \qquad (6)$$

Thus we have to define

$$[m^z] := [m^x] \pm [m^y]$$
$$[u_i^z] := [u_i^x] \pm [u_i^y], \quad (i = 1,2, \cdots, n) \qquad (7)$$

Multiplication:

To obtain a rule for multiplication of two generalized intervals, note that

$$[\breve{x}] \cdot [\breve{y}] = \{\breve{x} \cdot \breve{y}|\, \breve{x} \in [\breve{x}], \breve{y} \in [\breve{y}]\}$$
$$\subseteq [m^x] \cdot [m^y] + \sum_{i=1}^n \zeta_i ([m^x][u_i^y] + [m^y][u_i^x])$$
$$+ \sum_{i=1}^n \sum_{j=1}^n \zeta_i \zeta_j [u_i^x][u_j^y]$$

we shall choose to retain only linear terms in ζ_i $(i = 1,2, \ldots\ldots, n)$ although higher order terms could be kept.

Note that, we can take the absolute value (upper bound) for the term $\sum_{i=1}^n \sum_{j=1}^n \zeta_i \zeta_j [u_i^x][u_j^y]$ and add it to the midterm as follows.

$$[\breve{x}] \cdot [\breve{y}] \subseteq \left([m^x] \cdot [m^y] + \left| \sum_{i=1}^n \sum_{j=1}^n \zeta_i \zeta_j [u_i^x][u_j^y] \right| \right)$$
$$+ \sum_{i=1}^n \zeta_i ([m^x][u_i^y] + [m^y][u_i^x])$$
$$=: \ [\breve{z}] = [m^z] + \sum_{i=1}^n \zeta_i [u_i^z]$$

where

$$[m^z] := \left([m^x]\cdot[m^y] + \left|\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta_i\zeta_j\,[u_i^x][u_j^y]\right|\right) \quad (8)$$

and

$$[u_i^z] := [m^x][u_i^y] + [m^y][u_i^x] \quad (9)$$

Division:

For division of two generalized intervals, note that

$$\frac{[\check{x}]}{[\check{y}]} = \left\{\frac{\check{x}}{\check{y}} \mid \check{x}\in[\check{x}], \check{y}\in[\check{y}]\right\} \subseteq [m^z] + \sum_{i=1}^{n}\zeta_i[u_i^z]$$

with

$$[m^z] := \frac{[m^x]}{[m^y]} \quad (10)$$

and

$$[u_i^z] := \frac{[m^y][u_i^x]-[m^x][u_i^y]}{[m^y]\left([m^y]+[-1,1]\sum_{j=1}^{n} r_j u_j^y\right)} \quad (11)$$

Example 1. Consider

$$F = [x]\cdot[y] - [x]\cdot[y]$$

with $[x] = [1,2]$ *and* $[y] = [3,4]$.

Ordinary interval computation give

$F = [1,2].[3,4] - [1,2].[3,4]$
$= [3,8] - [3,8] = [-5,5]$

The exact result is [0,0] for any two intervals $[x]$ and $[y]$

Using generalized interval arithmetic with:

$[\check{x}] = [1.5,1.5] + [1,1]\zeta_1$, $\zeta_1\in$[-0.5,0.5]

$[\check{y}] = [3.5,3.5] + [1,1]\zeta_2$, $\zeta_2\in$[-0.5,0.5]

Now, by using the forms (8) and (9) give:

$$[\check{x}]\cdot[\check{y}] = [5.5,5.5] + [3.5,3.5]\zeta_1 + [1.5,1.5]\zeta_2$$

Then, the forms (6) and (7) give:

$$[\check{x}]\cdot[\check{y}] - [\check{x}]\cdot[\check{y}] = [0,0] + [0,0]\zeta_1 + [0,0]\zeta_2 = [0,0]$$

which is the exact result.

Example 2. [13] Consider

$F = \frac{[x_1]+[x_2]}{[x_1]-[x_2]}$ with $[x_1] = [1,2]$ and $[x_1] = [5,10]$.

Using (7) gives:

$$[\check{x}_1] + [\check{x}_2] = [9,9] + [1,1]\zeta_1 + [1,1]\zeta_2$$
$$[\check{x}_1] - [\check{x}_2] = [-6,-6] + [1,1]\zeta_1 + [-1,-1]\zeta_2$$

with $\zeta_1 \in [-0.5,0.5]$ and $\zeta_2 \in [-2.5,2.5]$

Using (10) and (11) we get:

$$F = -\frac{9}{6} + \left[-\frac{5}{6}, -\frac{5}{18}\right]\zeta_1 + \left[\frac{1}{18}, \frac{1}{6}\right]\zeta_2$$

which reduces to $[-2.334, -0.666]$.

The result $[-3.723, 0.7223]$ is obtained by Moore [13] using the mean value theorem.

Directed use of interval arithmetic yields $[-4, -0.666]$.

We obtain an exact result using interval arithmetic by rewriting F as $F = 1 + 2/([x_1]/[x_2] - 1)$ since each variable occurs only once. We find $F = [-2.334, -1.222]$.

Example 3. Consider

$F = [x] - \frac{10}{[x]+\frac{2}{[x]}}$, *with* $[x] = [1,3]$

By using ordinary interval computation, we get $F = [-5,1]$.

Using (6), (7) , (10) and (11) give:

$F = [-1.3333, -1.3333] + [1,1.95238]\zeta$, $\zeta \in [-1,1]$, which reduces to $[-3.2857143, 0.61904762]$.

While the mean-value form $F_{MN}([x])$ and the slope form $F_S([x])$ leads to the intervals [14] :

$F_{MN}([x])$= [*-5.1333, 2.4666*], $F_S([x])$= [*-3.6666,1*].

Example 4. Consider

$$f = (x_1\cdot x_2 - x_2)\cdot(x_1\cdot x_3 - x_3),$$

with $x_1 \in [5,10]$, $x_2 \in [1,2]$ and $x_3 \in [2,3]$

Using (6), (7) , (8) and (9) give:

$$F = [259,259] + [53.75,53.75]\zeta_1 + [113.75,113.75]\zeta_2 + [71.5,71.5]\zeta_3$$

where $\zeta_1 \in [-2.5,2.5]$, $\zeta_2 \in [-0.5,0.5]$ and $\zeta_1 \in [-0.5,0.5]$ which reduces *to* $F = [32,486]$.

Directed use of interval arithmetic yields $[21, 532]$.

We obtain an exact result using interval arithmetic by rewriting f as $f = x_2\cdot x_3\cdot(x_1-1)^2$ since each variable occurs only once. We find that $F = [72, 486]$.

4. Theoretical Background

In this section we give a brief summary of the theory of the enclosure method for our problem, in case of the system (1) involving affine-linear dependencies between the parameters.

Theorem 1. [23] *Consider parametric linear system (1), where A(p) and b(p) are defined by*

$$a_{ij}(p) := a_{ij}^{(0)} + \sum_{\gamma=1}^{k} p_\gamma a_{ij}^{(\gamma)}$$

$$b_i(p) := b_i^{(0)} + \sum_{\gamma=1}^{k} p_\gamma b_i^{(\gamma)}, \quad (i,j = 1,2,\cdots,n)$$

Let $R \in \mathbb{R}^n$, $[y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ *be given and define* $Z] \in \mathbb{IR}^n$ *and* $[C(p)] \in \mathbb{IR}^{n\times n}$ *by*

$$[Z] := R\cdot\left(b^{(0)} - A^{(0)}\tilde{x}\right) + \sum_{\gamma=1}^{k}[p_\gamma](Rb^{(\gamma)} - RA^{(\gamma)}\cdot\tilde{x})$$

$$[C(p)] := I - R\cdot A^{(0)} + \sum_{\gamma=1}^{k}[p_\gamma](R\cdot A^{(\gamma)})$$

where $A^{(0)} := \left(a_{ij}^{(0)}\right),\cdots,A^{(k)} := \left(a_{ij}^{(k)}\right) \in \mathbb{R}^{n\times n}$, $b^{(0)} := \left(b_i^{(0)}\right),\cdots,b^{(k)} := \left(b_i^{(k)}\right) \in \mathbb{R}^n$. *Define* $[v] \in \mathbb{IR}^n$ *by means of the following Single step method*

$$1 \le i \le n : [v_i] = \left\{\Diamond([Z] + [C]\cdot[u])\right\}_i \quad \textit{where}$$
$$[u] := ([v_1],\cdots,[v_{i-1}],[y_i],\cdots,[y_n])^\intercal$$

If $[v] \sqsubset [y]$[2], *then R and every matrix* $A(p), p \in [p]$ *is regular, and for every* $p \in [p]$ *the unique solution* $\hat{x} = A^{-1}(p)\cdot b(p)$ *of* $A(p)\cdot x = b(p)$ *satisfies* $\hat{x} \in \tilde{x} + [v]$.

[2] $\sqsubset$ is the inner inclusion relation

The above theorem generalized Theorem 4.8 from [15] by requiring of the range of $[C(p)]$ instead using an interval extension $[C([p])]$ [20].

5. Main Results

In this section, method for computing an outer solution for the system (1), in the general case, is suggested. The derivation of the method is based on the generalized interval arithmetic employed in section 3 in this paper above.

Let $f:[p] \subset \mathbb{R}^k \longrightarrow \mathbb{R}^k$ be a continuous function. The function $f(p)$ can be enclosed in the interval vector $[p]$ by the following linear interval form (generalized interval form)

$$[L_f(p)] := [m^f] + \sum_{\gamma=1}^{k} \zeta_\gamma [u_\gamma^f],\ p \in [p] \tag{12}$$

where $[m^f]$ and $[u_\gamma^f], (\gamma = 1,2,\cdots,k)$ are real intervals, and $\zeta_\gamma \in [-\mathrm{rad}([p_\gamma]), \mathrm{rad}([p_\gamma])]$. The form (12) can be determined in an automatic way using the method described in section 3. It has the inclusion property

$$f(p) \in [L_f(p)], p \in [p] \tag{13}$$

We assume that $a_{ij}(p)$ and $b_i(p), (i,j = 1,2,\cdots,n)$ in (2) are continuous functions. In accordance with (12), the corresponding linear interval forms are

$$[L_{ij}(p)] := [m^{a_{ij}}] + \sum_{\gamma=1}^{k} \zeta_\gamma [u_\gamma^{a_{ij}}]$$

$$[l_i(p)] := [m^{b_i}] + \sum_{\gamma=1}^{k} \zeta_\gamma [u_\gamma^{b_i}]$$

and have the inclusion property

$$a_{ij}(p) \in [L_{ij}(p)] := [m^{a_{ij}}] + \sum_{\gamma=1}^{k} \zeta_\gamma [u_\gamma^{a_{ij}}]$$

$$b_i(p) \in [l_i(p)] := [m^{b_i}] + \sum_{\gamma=1}^{k} \zeta_\gamma [u_\gamma^{b_i}]$$

Now, we can write a new parametric interval system as follows

$$[A(\zeta)] \cdot x = [b(\zeta)], \zeta \in [\zeta] \tag{14}$$

where and are defined by

$$\begin{aligned} [a_{ij}(\zeta)] &:= [a_{ij}^{(0)}] + \textstyle\sum_{\gamma=1}^{k} \zeta_\gamma [a_{ij}^{(\gamma)}], \\ [b_i(\zeta)] &:= [b_i^{(0)}] + \textstyle\sum_{\gamma=1}^{k} \zeta_\gamma [b_i^{(\gamma)}], \end{aligned} \tag{15}$$

The following theorem is a modification of theorem 1.

<u>Theorem 2.</u> *Consider parametric linear system (1), where A(p) and b(p) are defined by(2), Let* $[A(\zeta)] \in \mathbb{IR}^{n\times n}$ *and* $[b(\zeta)] \in \mathbb{IR}^n$ *be given by (15) with* $\zeta \in \mathbb{R}^k$*, and let* $R \in \mathbb{R}^n$, $[y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ *be given and define* $Z] \in \mathbb{IR}^n$ *and* $[C(\zeta)] \in \mathbb{IR}^{n\times n}$ *by*

$$[Z] := R \cdot \left([b^{(0)}] - [A^{(0)}]\tilde{x}\right) + \sum_{\gamma=1}^{k} [\zeta_\gamma](R \cdot [b^{(\gamma)}] - R \cdot [A^{(\gamma)}] \cdot \tilde{x})$$

$$[C(\zeta)] := I - R \cdot [A^{(0)}] + \sum_{\gamma=1}^{k} [\zeta_\gamma](R \cdot [A^{(\gamma)}])$$

where

$[A^{(0)}] := \left([a_{ij}^{(0)}]\right), \cdots, [A^{(k)}] := \left([a_{ij}^{(k)}]\right) \in \mathbb{IR}^{n\times n}$,

$[b^{(0)}] := \left([b_i^{(0)}]\right), \cdots, [b^{(k)}] := \left([b_i^{(k)}]\right) \in \mathbb{R}^n$

Define $[v] \in \mathbb{IR}^n$ *by means of the following Single step method*

$$1 \le i \le n : [v_i] = \left\{\Diamond\{[Z] + [C] \cdot [u]\}\right\}_i \quad \textit{where}$$

$$[u] := ([v_1], \cdots, [v_{i-1}], [y_i], \cdots, [y_n])^\intercal$$

If $[v] \sqsubset [y]$[3]*, then R and every matrix* $A(\zeta) \in [A(\zeta)], \zeta \in [\zeta]$ *is regular, so every matrix* $A(p), p \in [p]$ *is regular, and for every* $p \in [p]$ *the unique solution* $\hat{x} = A^{-1}(p) \cdot b(p)$ *of* $A(p) \cdot x = b(p)$ *satisfies* $\hat{x} \in \tilde{x} + [v]$.

Now, we give an algorithm for computing an outer solution for the system (1)

Algorithm 1: Parametric interval linear systems
1.*Initialization* $\hat{b} := \mathrm{mid}([b(\zeta)])$; $\hat{A} := \mathrm{mid}([A(\zeta)])$; *2.Computation of an approximate mid-point solution* $\tilde{x} = R \cdot \hat{b}$;($R \approx A^{-1}$) *3. Using The method described in section 2 to obtain the linear form for every element in the parametric matrix and right hand side vector.* *4.Computation of an enclosure [C]* $[C] := I - R \cdot [A^{(0)}] + \sum_{\gamma=1}^{k}[\zeta_\gamma](R \cdot [A^{(\gamma)}])$; *5. Computation of an enclosure [Z]* $[Z] := R \cdot \left([b^{(0)}] - [A^{(0)}]\tilde{x}\right) + \sum_{\gamma=1}^{k}[\zeta_\gamma](R \cdot [b^{(\gamma)}] - R \cdot [A^{(\gamma)}] \cdot \tilde{x})$; *6. Verification step* $[y] := [Z]$; max = 1; ***repeat*** $[yy] := [y]$; ***for*** $i = 1$ ***to*** n ***do*** $[y_i] = [Z_i] + [C(\mathrm{Row}(i))] \cdot [yy]$ max + +; ***until*** $[y] \sqsubset [yy]$ *or* max $\ge$ 10 *7.* ***if*** $[y] \sqsubset [yy]$ $x \in \tilde{x} + [y]$; ***else*** *{ Error;}* *8. end*

[3] $\sqsubset$ is the inner inclusion relation

6. Numerical and practical examples

The results are rounded outwardly to 8 digits accuracy.

Example 1: Consider the parametric linear system

$$\begin{pmatrix} -(p_1p_2+p_2) & p_1p_3 & p_2 \\ p_2p_4 & p_2^2 & 1 \\ p_1p_2 & p_3p_5+p_5 & p_2^2 \end{pmatrix} \cdot x = \begin{pmatrix} p_2p_3 \\ 1 \\ 1 \end{pmatrix}$$

$[p] = ([1,1.2],[2,2.2],[0.5,0.51],[0.39,0.4],[0.39,0.4])^{\intercal}$

Proposed method	Elaraby [28]
[-0.11831523,-0.07530551]	[-0.1186099,-.07514419]
[0.17114571, 0.20374365]	[0.17093864, 0.20434426]
[0.23211413, 0.27158007]	[0.23180722,0.27203779]

Example 2: Consider

$$\begin{pmatrix} -(p_1p_2+p_2) & p_1p_3 & p_2 \\ p_2p_4 & p_2^2 & 1 \\ p_1p_2 & p_3p_5+p_5 & p_2^2 \end{pmatrix} \cdot x = \begin{pmatrix} p_2p_3 \\ p_4p_5+p_4 \\ 1 \end{pmatrix},$$

$[p] = ([1,1.2],[2,2.2],[0.5,0.51],[0.39,0.4],[0.39,0.4])^{\intercal}$

Proposed method	Elaraby [28]
[-0.12264510,-.08131776]	[-0.12295217,-0.08115167]
[0.07593567, 0.09016943]	[0.07575593, 0.09048336]
[0.24435325, 0.29285135]	[0.24405491, 0.29335273]

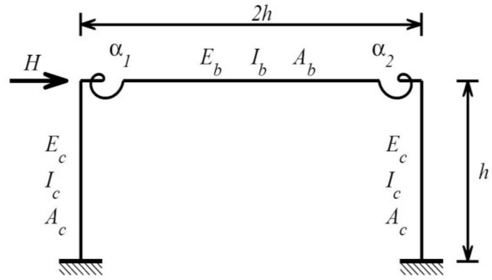
Practical example1 [9]: The new method will be illustrated with the following example. The elements of the linear system considered are

$$\begin{pmatrix} -(p_1+p_2)p_4 & p_2p_4 \\ p_5 & p_3p_5 \end{pmatrix} \cdot x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The interval vector $[p]$ is given by
$[p] =$
$([0.96,1.04],[1.92,2.08],[0.96,1.04],[0.48,0.52],[0.48,0.52])^{\intercal}$
Systems of the above type arise in robustness analysis of linear electric circuits [9] (for the example considered the radius is given as a 4% "tolerance" on the nominal parameter value $(1,2,1,0.5,0.5)^{\intercal}$).

Proposed method	Kolev's Method [9,17]	Elaraby[28]
[0.287555,0.513212]	[0.2844,0.5156]	[0.28704259,0.51295744]
[0.287555,0.513212]	[0.2844,0.5156]	[0.28704259,0.51295744]

Practical example2: Structural engineers use design codes formulated to consider uncertainty for both reinforced concrete and structural steel design. A simple one-bay structural steel frame (initially considered in [1]), is presented in the following figure.



The authors of [1] have applied conventional methods for analysis of frame structures to assemble a system of linear equations $K \cdot x = F$. In [1], the system has been presented as follows:

$$\begin{pmatrix}
\frac{12E_cI_c}{L_c^3}+\frac{A_bE_b}{L_b} & 0 & \frac{6E_cI_c}{L_c^2} & 0 & 0 & -\frac{A_bE_b}{L_b} & 0 & 0 \\
0 & \frac{12E_bI_b}{L_b^3}+\frac{A_cE_c}{L_c} & 0 & \frac{6E_bI_b}{L_b^2} & \frac{6E_bI_b}{L_b^2} & 0 & -\frac{12E_bI_b}{L_b^3} & 0 \\
\frac{6E_cI_c}{L_c^2} & 0 & \alpha+\frac{4E_cI_c}{L_c} & -\alpha & 0 & 0 & 0 & 0 \\
0 & \frac{6E_bI_b}{L_b^2} & -\alpha & \alpha+\frac{4E_bI_b}{L_b} & \frac{2E_bI_b}{L_b} & 0 & -\frac{6E_bI_b}{L_b^2} & 0 \\
0 & \frac{6E_bI_b}{L_b^2} & 0 & \frac{2E_bI_b}{L_b} & \alpha+\frac{4E_cI_c}{L_c} & 0 & -\frac{6E_bI_b}{L_b^2} & -\alpha \\
-\frac{A_bE_b}{L_b} & 0 & 0 & 0 & 0 & \frac{A_bE_b}{L_b}+\frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} \\
0 & -\frac{12E_bI_b}{L_b^3} & 0 & -\frac{6E_bI_b}{L_b^2} & -\frac{6E_bI_b}{L_b^2} & 0 & \frac{12E_bI_b}{L_b^3}+\frac{A_cE_c}{L_c} & -\frac{6E_bI_b}{L_b^2} \\
0 & 0 & 0 & 0 & -\alpha & \frac{6E_cI_c}{L_c^2} & -\frac{6E_bI_b}{L_b^2} & \alpha+\frac{4E_cI_c}{L_c}
\end{pmatrix} \cdot \begin{pmatrix} d2_x \\ d2_y \\ r2_z \\ r5_z \\ r6_z \\ d3_x \\ d3_y \\ r3_z \end{pmatrix} = \begin{pmatrix} H \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

whose elements are, in general, nonlinear functions of the following parameters: Material properties E_b, E_c , cross sectional properties I_b, I_c, A_b, A_c , lengths L_b, L_c, and the joint stiffness α. The right hand side vector $F = (H,0,0,0,0,0,0,0)^{\intercal}$ in this example is considered to depend only on the applied loading H. Table 1 will show the typical nominal parameter values and the corresponding worst case uncertainties as proposed in [1] .

Table 1 Parameters involved in the steel frame example, their nominal values, and worst case uncertainties

Parameters		Nominal value	Uncertainty
Young modulus	E_b	$29 \cdot 10^6$ lbs/in^2	±348*10^4(±12%)
	E_c	$29 \cdot 10^6$ lbs/in^2	±348*10^4(±12%)
Second moment	I_b	510 in^4	±51(±10%)
	I_c	272 in^4	±27.2(±10%)
Area	A_b	10.3 in^2	±1.03(±10%)
	A_c	14.4 in^2	±1.44(±10%)
External forces	H	5305.5 lbs	±2203.5(±41.6%)
Joint stiffness	α	2.77461*10^{8}lb-in/rad	±1.26504*10^8 (±45.6%)
Length	L_b	288 in	
	L_c	144 in	

In [1] all the parameters, except the lengths, are considered to be uncertain and varying within given intervals. Replacing L_b and L_c with their nominal values will give the following parametric interval linear system

$$K(p) \cdot x = F(p) \qquad (16)$$

where the vector of the uncertain parameters is $p = (E_b, E_c, I_b, I_c, A_b, A_c, \alpha, H)^{\intercal}$, the right hand side vector is $F(p) = (H, 0,0,0,0,0,0,0)^{\intercal}$ and the parametric matrix *K(p)* is

$$\begin{pmatrix} \frac{E_cI_c}{248832}+\frac{A_bE_b}{288} & 0 & \frac{E_cI_c}{3456} & 0 & 0 & -\frac{A_bE_b}{288} & 0 & 0 \\ 0 & \frac{12E_bI_b}{1990656}+\frac{A_cE_c}{144} & 0 & \frac{E_bI_b}{13824} & \frac{E_bI_b}{13824} & 0 & -\frac{E_bI_b}{1990656} & 0 \\ \frac{6E_cI_c}{3456} & 0 & \alpha+\frac{E_cI_c}{36} & -\alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{E_bI_b}{13824} & -\alpha & \alpha+\frac{E_bI_b}{72} & \frac{E_bI_b}{144} & 0 & -\frac{E_bI_b}{13824} & 0 \\ 0 & \frac{E_bI_b}{13824} & 0 & \frac{E_bI_b}{144} & \alpha+\frac{E_cI_c}{36} & 0 & -\frac{E_bI_b}{13824} & -\alpha \\ -\frac{A_bE_b}{288} & 0 & 0 & 0 & 0 & \frac{A_bE_b}{288}+\frac{E_cI_c}{248832} & 0 & \frac{E_cI_c}{3456} \\ 0 & -\frac{E_bI_b}{1990656} & 0 & -\frac{E_bI_b}{13824} & -\frac{E_bI_b}{13824} & 0 & \frac{E_bI_b}{1990656}+\frac{A_cE_c}{144} & -\frac{E_bI_b}{13824} \\ 0 & 0 & 0 & 0 & -\alpha & \frac{E_cI_c}{3456} & -\frac{E_bI_b}{13824} & \alpha+\frac{E_cI_c}{36} \end{pmatrix}$$

We will solve the system (16) by algorithm 1. The results will be compared with other methods based on the approach of [1]. In order to compare the results generated by our method and those generated by other methods, we strictly follow the structure system and the uncertainties for the parameters considered in [1]. Initially, the system (16) will be solved with parameter uncertainties which are 1% of the values presented in the last column of Table 1,
$A_b \in [10.289699, 10.313]$, $A_c \in [14.3856, 14.4144]$, $E_b \in [28965200, 29034800]$, $E_c \in [289652, 290348]$, $I_b \in [509.49, 510.51]$, $I_c \in [271.728, 272.272]$, $\alpha \in [276195900, 278726100]$, $H \in [5283.465, 5327.535]$.
A directed replacement approach, called naive interval approach, which does not take into account the dependencies between the parameters in solving practical problems. It is well-known that the solution of a naive interval system greatly overestimates the solution of the original parametric interval system. In [1], the naive interval results have been compared with the results obtained by the authors of [1].

Disp.	naive Interval [1]	Tight [1]	Popova [5]	Proposed method
$d2_x$	[0.09375783, 0.21337873]	[0.15237484, 0.15476814]	[0.15222223, 0.15431233]	[0.15220001, 0.15429121]
$d2_y$ e+3	[0.19060424, 0.47412283]	[0.32940418, 0.33533906]	[0.32377600, 0.32978730]	[0.32372754, 0.32974259]
$r2_z$ e+3	[-1.3531968, -0.57250484]	[-0.97085151, -0.95490139]	[-0.97197309, -0.95735919]	[-0.97161706,-0.95744291]
$r5_z$ e+3	[-0.6557609, -0.26414725]	[-0.4638112, -0.45611532]	[-0.46935397,-0.46200391]	[-0.46907740,-0.46214970]
$r6_z$ e+3	[-0.64100045, -0.2501251]	[-0.44930811, -0.4418354]	[-0.43060605, -0.42343378]	[-0.43018318,-0.42373289]
$d3_x$	[0.091230936, 0.21082444]	[0.14985048, 0.15221127]	[0.14968216, 0.15174482]	[0.14966080, 0.15172284]
$d3_y$ e+3	[-0.47412283,-0.19060424]	[-0.33533906, -0.32940418]	[-0.67739783,-0.66440928]	[-0.67730130,-0.66431021]
$r3_z$ e+3	[-1.3330326, -0.55323186]	[-0.95100335, -0.93531196]	[-0.93981876, -0.92572673]	[-0.93954242,-0.92573427]

6. Conclusion

The problem of solving parametric linear systems of equations whose elements are nonlinear function of interval parameter is very important in practical applications. Well-known classical methods, such as interval version of Gauss elimination, fail since they compute enclosure for the general solution set which is generally much larger than parametric solution set. In this work we reported newly developed software tools for solving parametric linear systems whose input data are non-linear functions of interval parameters. A simple method for determining an outer solution to the linear system considered has been suggested in section 5. An algorithm is presented and some numerical and practice examples are solved and compared with another methods.

The present approach is also applicable to other uncertainty theories which rely on interval arithmetic for computations, such as fuzzy set theory, random set theory, or probability bounds theory.

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Hassan Badry Mohamed A. El-Owny is Assistant Professor of Computer Science at Faculty of Science, Aswan University, Aswan, Egypt. His current position is Assistant Proffersor at Taif University, Taif, KSA. He received the PhD degree in ``Computer Science'' at University of Wuppertal(Germany). He is Aboard Editor of several international journals. He has published extensively in internationally refereed journals and conferences. His main research interests are: Scientific Computing, Computer Algebra, Interval Arithmatics, Parametric systems, Approximation of Uncertain Data and Software Tools with Result Verification