# Singularly Perturbed Quasilinear Boundary Value Problems With Interior shock Layer Behavior

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**Abstract**

Some singularly perturbed quasilinear boundary value problems are studied. Under certain conditions, by introducing a appropriate stretching transformation and constructing interior layer corrective terms, an asymptotic solution that is uniformly valid over the whole interval is obtained. This solution is shown to exhibit interior shock layer behavior at interior transition points.

**Keywords:** *Singularly perturbed; Quasilinear; Boundary value problems; shock layer; Composite expansion.*

## 1. Introduction

Consider the singularly perturbed quasilinear boundary value problem

$$\varepsilon y'' + f(x,y)y' + g(x,y) = 0,\ \ a < x < b\ , \quad \textbf{(1)}$$

$$y(a,\varepsilon) = A,\ y(b,\varepsilon) = B, \quad \textbf{(2)}$$

where $\varepsilon$ is a small positive parameter, $f$ and $g$ are infinitely differentiable functions
on an appropriate domain and $a < 0 < b$.

A number of author have studied problem (1),(2) under various assumptions, for example, see [1-4]. In particular, under the principal assumption that $f(0, y) = 0$ for all $y$ , i.e., that $x = 0$ is a turning point for the function $f$ , Howes **[5,6]** indicated that the existence of a solution of problem (1) ,(2) exhibiting interior layer behavior depends upon the behavior of the function $f$ near the turning point $x = 0$ . More precisely, if the function $f$ changes its algebraic sign in passing through zero, an interior shock layer appears at $x = 0$. If the function $f$ does not change its algebraic sign in passing through zero, then a boundary layer , but no transition layer , appears at one endpoint.

In this paper, we attempt to weaken the requirement that $f$ possesses a turning point at $x = 0$ (in a sense to be described below). By introducing a appropriate stretching transformation and constructing interior shock layer corrective terms**[7,8],** we can yield an $O(\varepsilon^m)$ approximate solution of problem (1) ,(2) as $\varepsilon \to 0$, for some positive integer $m$ uniformly valid on [a, b] . 2. Necessary conditions In this section, we derive a necessary condition for the existence of an interior shock layer solution of problem (1) ,(2) . Assume (I) There exist functions $u_L(x)$ and $u_R(x)$ of $C^2$on $[a, b]$ satisfying , the reduced problems

$$f(x,u)u' + g(x,u) = 0,\ \ u(a) = A \quad \textbf{(3)}$$

and

$$f(x,u)u' + g(x,u) = 0,\ \ u(b) = B \quad \textbf{(4)}$$

respectively , and $u_L(0) \neq u_R(0)$.

We seek an outer expansion in the form

$$U(x,\varepsilon) = \sum_{i=0}^{\infty} u_i(x)\varepsilon^i. \quad \textbf{(5)}$$

Substituting (5) into (1) and $y(a,\varepsilon) = A$, expanding

$$f(x,U) = f(x,u_0) + f_y(x,u_0)\sum_{i=0}^{\infty} u_i(x)\varepsilon^i + \frac{1}{2!} f_{yy}(x,u_0)(\sum_{i=1}^{\infty} u_i(x)\varepsilon^i)^2 + \cdots$$

and $g(x,U)$ (expanding as above replaces f with $g$) , and equating coefficients of like powers of $\varepsilon$ respectively, we obtain

$$f(x,u_0)u_o' + g(x,u_0) = 0, u_0(a) = A,$$

and

$$f(x,u_0)u_i' + [f_y(x,u_0)u_0' + g_y(x,u_0)]u_i = F_{i-1}, u_i(a) = 0 (i = 1,2,\ldots), \quad \textbf{(6)}$$

in which $u_0(x) = u_L(x)$ solves the reduced problem (3), and $F_{i-1}$ are determined functions of $u_0$, $u_{1,\ldots,}$ $u_{i-1}$ successively. Obviously each $u_i \in C^2[a,0]$, $(i = 1,2 \ldots)$ may be solved successively from terminal value problems (6). Also, the expansion (5) remains valid on [0, *b*] if we simply replace $u(a) = A$ with $u(b) = B$ in the calculations above, where $u_0(x) = u_R(x)$. Note that, generally, $u_i(0^-) \neq u_i(0^+)$ $(i = 1,2 \ldots)$in (5). Since $u_L(0) \neq u_R(0)$, to analyze the behavior of the solution in the transition layer, we need to stretch the neighborhood of $x = 0$. Thus, we let

$$\xi = \frac{x}{\varepsilon}.$$

and seek an inner expansion in the form

$$V(\xi, \varepsilon) = \sum_{i=0}^{\infty} v_i(\xi)\varepsilon^i;$$

Substituting

$$y = U(x, \varepsilon) + V(\xi, \varepsilon)$$

Into (1), yields

$$\ddot{V} + f(\xi\,\varepsilon, U + V)\dot{V} + \varepsilon[f(\xi\,\varepsilon, U + V) - f\xi\,\varepsilon,UU+\varepsilon[g\xi\,\varepsilon,U+V-g\xi\,\varepsilon,U=0$$

or

$$\ddot{V} + f(\xi\,\varepsilon, U + V)\dot{V} + \varepsilon[f_y(\xi\,\varepsilon, U + \theta_1 V)\acute{U} + g_y(\xi\,\varepsilon, U + \theta_2 V)]V = 0, \quad \textbf{(7)}$$

Where $\dot{V} = \frac{dV}{d\xi}$, $\ddot{V} = \frac{d^2V}{d\xi^2}$, and $0<\theta_1, \theta_2<1$,. Expanding

$$f(\xi\,\varepsilon, U + V) = f(0, u_0(0) + V_0) + f_x(0, u_0(0) + V_0)\xi\,\varepsilon$$

$$+f_y(0, u_0(0) + V_0)\left(\sum_{i=0}^{\infty} u_i\varepsilon^i - u_0(0) + \sum_{i=1}^{\infty} v_i\varepsilon^i\right) + \cdots,$$

$$f_y(\xi\,\varepsilon, U + \theta_1 V) = f_y\big(0, u_0(0)\big) + f_{yx}\big(0, u_0(0)\big)\xi\,\varepsilon$$

$$+f_{yy}\big(0, u_0(0)\big)\left(\sum_{i=0}^{\infty} u_i\varepsilon^i - u_0(0) + \theta_1\sum_{i=0}^{\infty} v_i\varepsilon^i\right) + \cdots$$

and $g_y(\xi\,\varepsilon, U + \theta_2 V)$ (expanding as above replace $f\ by\ g$ and $\theta_1$ by $\theta_2$), and equating coefficients of like powers of $\varepsilon$ in (7), we obtain

$$\ddot{v}_0 + f(0, u_0(0) + v_0)\dot{v}_0 = 0 \quad (\mathbf{8})$$

And

$$\ddot{v}_i + f(0, u_0(0) + v_0)\dot{v}_i + f_y(0, u_0(0) + v_0)\dot{v}_0 v_i = G_{i-1}(i = 1,2, \ldots), \quad (\mathbf{9})$$

Where $G_{i-1}$ are determined functions of $v_0, v_1, \ldots, v_{i-1}$ successively. In what follows, without loss of generality, we take $u_0(0) = u_L(0)$. Since $v_0(\xi)$is the dominant term of correction which must satisfy.

$$\lim_{\xi\to-\infty} v_0(\xi) = 0,\ \lim_{\xi\to+\infty} v_0(\xi) = u_R(0) - u_L(0)$$

and

$$\lim_{\xi\to\infty} \dot{v}_0(\xi) = 0.$$

Consider $v_0 \to 0\ and\ \dot{v}_0 \to 0\ as\ \xi \to -\infty$, it follows from (8) that

$$\dot{v}_0 = F(v_0), F(v_0) = -\int_{u_L(0)}^{u_L(0)+v_0} f(0, w)dw.$$

Consequently, $v_0(\xi)$ may be given implicitly by

$$\xi = \int_{v_0(0)}^{v_0} \frac{dw}{F(w)}\ , \quad \textbf{(10)}$$

Hence, we see that $v_0(\xi)$ is monotone strictly increasing whenever $F(v_0) > 0$ when $u_R(0) > u_L(0)$, while $v_0(\xi)$ is monotone strictly decreasing whenever $F(v_0) < 0$ when $u_R(0) < u_L(0)$. Putting both together , We have

$$[u_R(0) - u_L(0)]F(v_0) > 0$$

Or

$$[u_R(0) - u_L(0)]\ \int_{u_L(0)}^{z} f(0, w)dw < 0\ , \quad \textbf{(11)}$$

For all $z$ between $u_L(0)\ and\ u_R(0)$.

Also, consider $v_0 \to u_R(0) - u_L(0)\ and\ \dot{v}_0 \to 0\ as\ \xi \to +\infty, it\ follows\ that$

$$\int_{u_L(0)}^{u_R(0)} f(0, w)dw = 0\ , \quad (\mathbf{12})$$

As before, each $v_i(\xi)(i = 1,2, \ldots)$ in (9) with conditions

$$\lim_{\xi\to-\infty} v_i(\xi)=0, \lim_{\xi\to+\infty} v_i(\xi)=u_i(0^+)-u_i(0^-)$$

and

$$\lim_{\xi\to\infty}\dot{v}_0(\xi)=0.$$

may be determined successively.

We have thus far formed a composite expansion of problem (1),(2) on [a, b] as Follows

$$y(x,\varepsilon)=\sum_{i=0}^{\infty}u_i(x)\varepsilon^i+\sum_{i=0}^{\infty}v_i\left(\frac{x}{\varepsilon}\right)\varepsilon^i. \qquad (\mathbf{13})$$

Clearly, inequality (11) (or write as $[\,u_R(0)-u_L(0)]\int_{u_R(0)}^{z}f(0,w)dw<0$ ), condition (12) and the expansion (13) corresponding to $u_R(0)$ remain valid on [a, b] if we simply replace $u_L(0)$ with $u_R(0)$ in the discussion above. It should be noted that, in this case, we would require

$$\lim_{\xi\to+\infty} v_0(\xi)=0, \lim_{\xi\to-\infty} v_0(\xi)=u_L(0)-u_R(0)$$

and

$$\lim_{\xi\to\infty}\dot{v}_0(\xi)=0.$$

We have proven

**Theorem1** Under hypothesis (I), if a solution of problem (1),(2) which exhibits interior shock layer behavior at $x=0$ , then inequality (11) and condition (12) are satisfied.

## 2. Sufficient conditions

We can now give sufficient conditions for the existence of a solution of problem (1),(2) exhibiting interior shock layer behavior near $x=0$ .

**Theorem 2** Assume that (I) holds , and that

$$(II)[u_R(0)-u_L(0)]f\big(0,u_L(0)\big)<0 \; and \; [u_R(0)-u_L(0)]f\big(0,u_R(0)\big)>0;$$

(III) There exist some constant $k>0$ so that

$$f_y\big(0,u_0(0)\big)\acute{u}_0(0)+g_y\big(0,u_0(0)\big)\le -k$$

$$for \;\; u_0(0)=u_L(0) \; or \;\; u_0(0)=u_R(0);$$

(IV) the inequality (11) and the condition (12) are satisfied. Then, for sufficient small, problem (1),(2) has a solution $y=y(x,\varepsilon)$ with

$$y(x,\varepsilon)=\sum_{i=0}^{m}u_i(x)\varepsilon^i+\sum_{i=0}^{m}v_i\left(\frac{x}{\varepsilon}\right)\varepsilon^i+O(\varepsilon^{m+1}).$$

as $\varepsilon\to 0$ , for some $m$ uniformly on [a, b].This solution exhibits shock layer behavior at $x=0$.

**Proof.** We consider only the case that $u_R(0)>u_L(0)$ and $u_0(0)=u_L(0)$ in (III) Since the other case is similar.

First, we need to show that given any $\tau\in(0,\sigma), v_0(\xi)$ satisfies

$$v_0(\xi)=O\big(e^{(\sigma_1-\tau)\xi}\big)(\xi\to-\infty), \qquad (\mathbf{14})$$

and

$$u_R(0)-u_L(0)-v_0(\xi)=O\big(e^{-(\sigma_2-\tau)\xi}\big)\quad(\xi\to+\infty)\quad, \qquad (15)$$

where $\sigma_1=-f\big(0,u_L(0)\big),\sigma_2=-f\big(0,u_R(0)\big)\; and$

$$\sigma=\min\{\sigma_1,\sigma_2\}.$$

Since given such $\tau>0$ , by the continuity of $f$ with respect to , we can choose a number $w_\tau\in\big(0,v_0(0)\big)$ so that

$$-f(0,u_L(0)+s)\ge(\sigma_1-\tau)$$

$for\; 0\le s\le w_\tau$.Consequently,

$$F(v_0)=-\int_{u_L(0)}^{u_L(0)+v_0}f(0,w)dw\ge(\sigma_1-\tau)\,v_0$$

$for\;\; 0\le v_0\le w_\tau$ . Again, note that $F(v_0)$ is bounded on $[w_\tau,v_0(0)]$ , a positive constant $C_\tau$ is chosen so that

$$\ln\frac{C_\tau}{w_\tau}\ge(\sigma_1-\tau)\int_{w_\tau}^{v_0(0)}\frac{dw}{F(w)}.$$

Hence, from (10) we have, for $v_0<w_\tau$

$$-\xi=\int_{v_0}^{v_0(0)}\frac{dw}{F(w)}=\int_{v_0}^{w_\tau}\frac{dw}{F(w)}+\int_{w_\tau}^{v_0(0)}\frac{dw}{F(w)}$$

$$\leq \frac{1}{(\sigma_1-\tau)}(\ln\frac{w_\tau}{v_0}+\ln\frac{C_\tau}{w_\tau}) = \frac{1}{(\sigma_1-\tau)}\ln\frac{C_\tau}{v_0}.$$

Therefore $v_0 \leq C_\tau e^{(\sigma_1-\tau)\xi}$. On the other hand if $v_0 \geq w_\tau$ we take $\widetilde{C_\tau} = \frac{C_\tau}{w_\tau}v_0(0)$,
then

$$-\xi = \int_{v_0}^{v_0(0)}\frac{dw}{F(w)} \leq \int_{w_\tau}^{v_0(0)}\frac{dw}{F(w)} \leq \frac{1}{(\sigma_1-\tau)}\ln\frac{C_\tau}{w_\tau} \leq \frac{1}{(\sigma_1-\tau)}\,ln\frac{\widetilde{C_\tau}}{v_0}.$$

Also, the results is $v_0 \leq \widetilde{C_\tau}e^{(\sigma_1-\tau)\xi}$.So the estimate (14) is true.

In a similar way, we can conclude that (15) is valid, one need merely rewrite
$F(v_0)$ as

$$F(v_0) = \int_{u_L(0)+v_0}^{u_R(0)} f(0,w)dw$$

using condition (12). Further.
fix some $\tau \in (0,\sigma)$ and rewrite
$\sigma_1-\tau = \lambda_0$, $\sigma_2-\tau = \mu_0$. It follows (9) that
$v_i(\xi) = O(e^{\lambda_i\xi})(\xi \to -\infty)$ and $v_i(\xi) - u_i(0^+) + u_i(0^-) = O(e^{-\mu_i\xi})(\xi \to +\infty)$,
Where $\lambda_i > 0, \mu_i > 0$ and $\lambda_i \geq \lambda_j$, $\mu_i \geq \mu_j$ for $j > i(i = 1,2,\dots)$.

Next, given any positive integer *m*, define

$$\alpha(x,\varepsilon) = \sum_{i=0}^{m} u_i(x)\varepsilon^i + \sum_{i=0}^{m} v_i\left(\frac{x}{\varepsilon}\right)\varepsilon^i - l\varepsilon^{m+1},$$

$$\beta(x,\varepsilon) = \sum_{i=0}^{m} u_i(x)\varepsilon^i + \sum_{i=0}^{m} v_i\left(\frac{x}{\varepsilon}\right)\varepsilon^i + l\varepsilon^{m+1};$$

From the construction of $v_i(i = 0,1,2,\dots,m)$ and note that $\dot{v}_0$ and $v_i(i = 1,2,\dots,m)$ are exponentially small terms, we have, for $\varepsilon$ sufficiently small,

$$\begin{aligned}\varepsilon[\varepsilon\alpha'' + f(x,\alpha)\alpha' + g(x,\alpha)] &= \ddot{v}_0 + f\big(0,u_L(0)\big) + v_0)\dot{v}_0 \\ &+ \sum_{i=1}^{m}[\ddot{v}_i + f\big(0,u_L(0)\big) + v_0)\dot{v}_i + f_y(0,u_L(0) + v_0)\dot{v}_0 v_i - G_{i-1}]\varepsilon^i \\ &-\big[f_y\big(0,u_L(0)\big)u_L'(0) + g_y\big(0,u_L(0)\big)\big]l\varepsilon^{m+2} + O(\varepsilon^{m+2})\end{aligned}$$

Thus, there exist some constant $k > 0$ such that

$$\varepsilon\alpha'' + f(x,\alpha)\alpha' + g(x,\alpha) \geq kl\varepsilon^{m+1} - K\varepsilon^{m+1} \geq 0$$

if $l$ is chosen so that $kl \geq K$, where we have made the hypothesis of (III).
Proceeding similarly, we have

$$\varepsilon\beta'' + f(x,\beta)\beta' + g(x,\beta) \leq 0.$$

Moreover, clearly, $\alpha(x,\varepsilon) \leq \beta(x,\varepsilon)\ for\ a < x < b, \alpha(a,\varepsilon) \leq A \leq \beta(a,\varepsilon)\ and$
$\alpha(b,\varepsilon) \leq B \leq \beta(b,\varepsilon)$. By the theory of differential inequalities [**9**] , we conclude that
there is a solution $y = y(x,\varepsilon)$ of problem (1),(2) such that $\alpha(x,\varepsilon) \leq y(x,\varepsilon) \leq \beta(x,\varepsilon)$
on [a, b]. More precisely,

$$y(x,\varepsilon) = \sum_{i=0}^{m} u_i(x)\varepsilon^i + \sum_{i=0}^{m} v_i\left(\frac{x}{\varepsilon}\right)\varepsilon^i + O(\varepsilon^{m+1}),$$

as $\varepsilon \to 0$, for some positive integer $m$ uniformly on [a, b]. Since $\alpha(x,\varepsilon)$ and $\beta(x,\varepsilon)$ both converge to $u_0(x)$ for $x \in [a,0) \cup (0,b]\ and\ to\ s\ for\ x = 0\ as\ \varepsilon \to 0$, the solution $y(x,\varepsilon)$ must behave in the same way. Again since $u_L(0) \neq u_R(0)$ and s between $u_L(0)\ and\ u_R(0)$ , we say that $y(x,\varepsilon)$ exhibits interior shock layer behavior at $x = 0$. This completes the proof of Theorem 2.
As an application of Theorem 2, we consider the boundary value problem

$$\varepsilon y'' + (x - cosy)y' + (x+1)y(y-\pi) = 0, \quad \textbf{(16)}$$

$$y(-1,\varepsilon) = 0, \quad y(1,\varepsilon) = \pi. \qquad (17)$$

Clearly, $u_L(x) \equiv 0$ *and* $u_R(x) \equiv 0$ solve the reduced problems

$$(x - cosu)u' + (x+1)u(u-\pi) = 0, u(-1) = 0$$

and

$$(x - cosu)u' + (x+1)u(u-\pi) = 0, u(1) = \pi$$

respectively. Now

$$u_R(0) > u_L(0), f\big(0, u_L(0)\big) = -1 < 0,$$

$$f\big(0, u_R(0)\big) = 1 > 0$$

$$\int_0^z f(0,w)dw = -\sin z < 0 \; for \; 0 < z < \pi \;\; and \; \int_0^\pi f(0,w)dw = 0.$$

Moreover, $f_y\big(0, u_L(0)\big) = 0$ *and* $g_y\big(0, u_L(0)\big) = -\pi < 0$. Thus ,all the conditions are satisfied .

We conclude that problem (16),(17) has a solution $y = (x,\varepsilon)$ exhibiting shock layer behavior at $x = 0$. More precisely, it follows from (10) that

$$\xi = \int_{v_0(0)}^{v_o} \frac{dw}{sinw} = \ln \tan \frac{v_0}{2}$$

or

$$v_0 = 2 \arctan e^{\xi}$$

If $v_0(0) = \frac{\pi}{2}$ is chosen. So we have

$$y(x,\varepsilon) = 2 \arctan e^{\frac{x}{\varepsilon}} + O(\varepsilon)$$

as $\varepsilon \to 0$, uniformly on [-1,1].Continuing the process, we can obtain a higher order approximation of (16) and (17). The details are omitted.

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