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The steady-state solution analysis for the degenerate nonlocal parabolic equation

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Abstract

In this paper, we investigate the steady-state solution for the degenerate nonlocal parabolic equation.we prove that the equation corresponds a unique steady-state solution under certain conditions.

Keywords: *Parabolic Equation, The Steady-State Solution, Ohmic Heating, Nonlocal Parabolic Equation.*

1. Introduction

In this short paper,we investigate the steady-state solution for the following parabolic equation with nonlocal and degenerate source, i.e.,

$$u_t - \nabla \cdot (u^3 \nabla u) = \frac{\lambda \exp(-u^4)}{(\int_\Omega \exp(-u^4)dx)^2},$$

(1)

where $x \in \Omega \subset R^2$ and $t > 0$.

With the homogeneous Dirichlet boundary conditions as

$$u(x,t) = 0, \qquad x \in \partial\Omega, t > 0$$

(2)

and

$$u(x,0) = u_0(x) > 0. \qquad x \in \Omega$$

(3)

where $\lambda > 0$ and $\Omega = \{x \in R^2 : 0 < \rho < |x| < R\}$.

In the past several decades, many physical phenomena have been formulated into nonlocal mathematical models.Let us mention,for instance,Lacey [1,2] has obtained the nonlocal parabolic equations

$$\begin{cases} u_t - \nabla u = \dfrac{\lambda f(u)}{(\int_\Omega f(u)dx)^2}, x \in \Omega, t > 0, \\ u = 0, x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), x \in \Omega \end{cases}$$

(4)

Where u is the temperature of the heated object.

Eq.(4), as a kind of Ohmic heating model,which comes from the more general parabolic-elliptic equations

$$\begin{cases} u_t - \nabla \cdot (\kappa(u)\nabla u) = \sigma(u)|\nabla \phi|^2, x \in \Omega, t > 0, \\ \nabla \cdot (\sigma(u)\nabla \phi) = 0, x \in \Omega, t > 0. \end{cases}$$

(5)

Where ϕ is the voltage at the ends of the conductor.

These two equations were studied in [1,2,3] and [4-9] respectively.

Investigation on Eq.(1-3) Mainly include three problems: the existence and uniqueness of the steady-state solution , the rate of blow-up and asymptotic analysis for the equations.

The work of this paper is motivated by the steady-state source problem

$$\nabla \cdot (w^3 \nabla w) + \frac{\lambda \exp(-w^4)}{(\int_\Omega \exp(-w^4)dx)^2} = 0,$$

(6)

where $x \in \Omega$, $w = 0$ and $x \in \partial\Omega$.

The existence of solution to the problem (6) has closely Relationship with the following problem

$$\nabla \cdot (w^3 \nabla w) + \mu \exp(-w^4) = 0,$$

(7)

where $x \in \Omega, w = 0$ and $x \in \partial\Omega$.

Here we set $\mu \geq 0$ and $\lambda(\mu) = \mu(\int_\Omega \exp(-w^4)dx)^2$.

2. Main Results

The main result of this paper reads as follows:

Theorem 2.1

Assume that $\Omega = \{x \in R^2 : 0 < \rho < |x| < R\}$ and that $\lambda^* = |\partial\Omega|^2/2$, we have

(i) If $0 < \lambda < \lambda^*$, the problem (6) corresponds a solution at least.

(ii) If $\lambda \geq \lambda^*$, the problem (6) have no solution.

Theorem 2.2

Assume that $\Omega = \{x \in R^2 : 0 < \rho < |x| < R\}$ and that $\lambda^* = |\partial\Omega|^2/2$. If $0 < \lambda < \lambda^*$, then we have the problem (6) corresponds a unique steady-state solution.

3. Proof of Theorem 2.1 and Theorem 2.2

First of all, we prepare some definitions, notations which will be needed in the proof of our results.

We assume $w(r;\mu)$ is radially symmetric, Let $w(r;\mu)$ be a solution of (7). By the maximum principle, form (7), we have

$$(w^3 w_r)_r + \frac{1}{r} w^3 w_r + \mu \exp(-w^4) = 0, \rho < r < R;$$
$$w(\rho) = w(R) = 0 \tag{8}$$

Which implies

$$-(rw^3 w_r)_r = \mu r \exp(-w^4), \rho < r < R, \tag{9}$$

and

$$-((rw^3 w_r)^2)_r = \mu r^2 w^3 w_r \exp(-w^4), \rho < r < R, \tag{10}$$

From (9), we obtain a unique solution

$$r_0 = r_0(\mu) \in (\rho, R)$$

Such that

$$w(r_0;\mu) = \max_{[\upsilon,R]} w(r;\mu) = M(\mu).$$

Integrating both sides of (9) and (10) over (r, r_0), we have, for $\rho < r < R$,

$$\frac{1}{2}(rw^3 w_r)^2 = \frac{1}{4}\mu(r^2 e^{-w^4} - r_0^2 e^{-M^4}) + \frac{1}{2} rw^3 w_r,$$

This equality infers that

$$(\frac{1}{2} - rw^3 w_r)^2 = \frac{1}{4} + \frac{1}{2}\mu(r^2 e^{-w^4} - r_0^2 e^{-M^4}), \tag{11}$$

Set $L_1(\mu) = \lim_{r \to \rho+} rw^3 w_r$ and $L_2(\mu) = \lim_{r \to R-} rw^3 w_r$.

From (11), we have

$$L_1(\mu) = \begin{cases} \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{2}\mu(\rho^2 - r_0^2 e^{-M^4})}, L_1(\mu) \leq \frac{1}{2}, \\ \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{2}\mu(\rho^2 - r_0^2 e^{-M^4})}, L_1(\mu) > \frac{1}{2}, \end{cases} \tag{12}$$

and

$$L_2(\mu) = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{2}\mu(R^2 - r_0^2 e^{-M^4})} \tag{13}$$

By the definition of $\lambda(\mu)$, it holds that

$$\lambda(\mu) = \frac{4\pi^2}{\mu}(L_1(\mu) - L_2(\mu))$$

$$\text{Set } \Gamma(\mu) = \frac{1}{2\pi}\sqrt{\lambda(\mu)} = \frac{1}{\sqrt{\mu}}(L_1(\mu) - L_2(\mu)), \tag{14}$$

Combining (12) and (13), we have

$$\Gamma(\mu) = \begin{cases} \sqrt{\frac{1}{4\mu} + \frac{1}{2}y_1} - \sqrt{\frac{1}{4\mu} + \frac{1}{2}y_2}, L_1(\mu) \leq \frac{1}{2} \\ \sqrt{\frac{1}{4\mu} + \frac{1}{2}y_1} + \sqrt{\frac{1}{4\mu} + \frac{1}{2}y_2}), L_1(\mu) > \frac{1}{2} \end{cases} \tag{15}$$

where $y_1 = R^2 - r_0^2 e^{-M^4}$ and $y_2 = \rho^2 - r_0^2 e^{-M^4}$.

Through a series of preparations, we derive a fact of $\Gamma(\mu)$.

Lemma 1

(i) If $L_1(\mu) \leq \frac{1}{2}$, hence $\Gamma(\mu) < (R+\rho)/\sqrt{2}$.

(ii) If $L_1(\mu) > \frac{1}{2}$,

hence $\Gamma(\mu) < (R+\rho)/\sqrt{2} \Leftrightarrow \mu r_0^2 e^{-M^4} > \frac{1}{2}$,

and $\Gamma(\mu) = (R+\rho)/\sqrt{2} \Leftrightarrow \mu r_0^2 e^{-M^4} = \frac{1}{2}$.

Through a series of calculation yields, we can prove the lemma 1.Here we omit the proof of lemma 1 because of the length of the article.

Proof of Theorem 2.1

Proof.Set $y=\frac{1}{2}-rw^3w_r$. Combining (9) and (11),we have

$$\frac{1}{2}r\frac{dy}{dr}=y^2+\frac{1}{2}\mu r_0^2e^{-M^4}-\frac{1}{4}.$$

(16)

In the case of $\mu=\mu_1$,we then obtain

$$\frac{1}{2}r\frac{dy}{dr}=y^2$$

(17)

Now according to (9),(13) and Lemma 1,we see that there exists $r_1>\rho$, such that

$$r_1w^3(r_1;\mu_1)w_r(r_1;\mu_1)=\frac{1}{2}.$$

Integrating both sides of (17) over (r,r_0), we have,for $r_1<r<r_0$,

$$\frac{1}{1/2-rw^3w_r}=2+2(\ln r_0-\ln r),$$

This is a contradiction of the equation for $r\to r_1$ and we then complete the Proof of Theorem 2.1.

In order to prove Theorem 2.2, We need to derive a fact of the following two problems.

Lemma2

Denote $\mu r_0^2(\mu)e^{-M^4(\mu)}=1/2 \quad , \quad \mu>0,$

(18)

We then have a unique solution

$$\mu_I=\frac{(R-\rho)^2}{2\rho^2R^2(\ln R-\ln\rho)^2}$$

(19)

Proof.From (12),we have $\lim_{\mu\to\infty}L_1(\mu)=\infty$. Now according to theorem 2.1 and Lemma 1(ii),we obtain $\mu r_0^2(\mu)e^{-M^4(\mu)}>1/2$ and $L_1(\mu)<1/2.$ which implies that there exists μ_I satisfies (18), Integrating both sides of (17) over (ρ,r_0) and (r_0,R) respectively,we have

$$\frac{1}{1/2-L_1(\mu_I)}-2=2(\ln r_0-\ln\rho),$$

(20)

and $$2-\frac{1}{1/2-L_2(\mu_I)}=2(\ln R-\ln r_0).$$

(21)

Using (12) and (13),we infer that

$$\begin{cases}1/2-L_1(\mu_I)=\sqrt{\frac{1}{2}\mu_I\rho},\\ 1/2-L_2(\mu_I)=\sqrt{\frac{1}{2}\mu_I R}.\end{cases}$$

(22)

Combining (20) to (22),we obtain a unique solution

$$\mu_I=\frac{(R-\rho)^2}{2\rho^2R^2(\ln R-\ln\rho)^2}.$$

Lemma3

Denote $$L_1(\mu)=\frac{1}{2},\mu>0,$$

(23)

We then have a unique solution

$$\mu_{II}=\frac{(\arctan\frac{\sqrt{R^2-\rho^2}}{\rho})^2}{2\rho^2(\ln R-\ln\rho)^2}.$$

(24)

Proof.Similar to the proof of Lemma 2,we have

$$\frac{1}{2}r\frac{dy}{dr}=y^2+\frac{1}{2}\mu_{II}\rho^2.$$

(25)

Integrating both sides of (17) over (ρ,r_0) and (r_0,R) respectively,we have

$$\frac{1}{\sqrt{2\mu_{II}\rho^2}}\arctan\frac{1}{\sqrt{2\mu_{II}\rho^2}}=\ln r_0-\ln\rho \quad (26)$$

and $$\frac{1}{\sqrt{2\mu_{II}\rho^2}}\arctan\frac{1-2L_2(\mu_{II})}{\sqrt{2\mu_{II}\rho^2}}+\frac{1}{\sqrt{2\mu_{II}\rho^2}}\arctan\frac{1}{\sqrt{2\mu_{II}\rho^2}}=\ln R-\ln r_0 \quad .$$

(27)

From (13),we have

$$1/2-L_2(\mu_{II})=\sqrt{\frac{1}{2}\mu_{II}(R^2-\rho^2)}.$$

(28)

Combining (26) to (28),we obtain a unique solution

$$\mu_{II}=\frac{(\arctan\frac{\sqrt{R^2-\rho^2}}{\rho})^2}{2\rho^2(\ln R-\ln\rho)^2}.$$

Proof of Theorem 2.2

Proof.Set $$G(\mu)=\frac{1}{4\mu}-\frac{1}{2}r_0^2e^{-M^4}$$

(29)

in view of(16),we observe

$$\frac{1}{2}r\frac{dy}{dr}=y^2-\mu G(\mu)$$

(30)

We have three steps to prove Theorem 2.2.

Step 1 If $0<\mu<\mu_I$,Using lemma 2,it holds that

$$\mu r_0^2(\mu)e^{-M^4(\mu)}<1/2,$$

which implies $G(\mu)>0$. Using lemma 1 and Theorem 2.1,we obtain $L_1(\mu)<1/2$.

Integrating both sides of (30) over (ρ,r_0) and (r_0,R) respectively,we have

$$\frac{1}{\sqrt{\mu G(\mu)}}(\ln\frac{1-2\sqrt{\mu G(\mu)}}{1+2\sqrt{\mu G(\mu)}}-\ln\frac{1-2L_1(\mu)-2\sqrt{\mu G(\mu)}}{1-2L_1(\mu)+2\sqrt{\mu G(\mu)}})=4(\ln r_0-\ln\rho) .$$

(31)

and

$$\frac{1}{\sqrt{\mu G(\mu)}}(\ln\frac{1-2L_2(\mu)-2\sqrt{\mu G(\mu)}}{1-2L_2(\mu)+2\sqrt{\mu G(\mu)}}-\ln\frac{1-2\sqrt{\mu G(\mu)}}{1+2\sqrt{\mu G(\mu)}})=4(\ln R-\ln r_0) .$$

(32)

From (12) and (13),we obtain

$$\begin{cases}1/2-L_1(\mu_I)=\sqrt{\frac{1}{2}\mu\rho^2+\mu G(\mu)},\\ 1/2-L_2(\mu_I)=\sqrt{\frac{1}{2}\mu\rho^2+\mu G(\mu)}.\end{cases}$$

(33)

Combining (31) to (33),we then have

$$\frac{1}{\sqrt{G(\mu)}}(\ln\frac{\sqrt{R^2/2+G(\mu)}-\sqrt{G(\mu)}}{\sqrt{R^2/2+G(\mu)}+\sqrt{G(\mu)}}-\ln\frac{\sqrt{\rho^2/2+G(\mu)}-\sqrt{G(\mu)}}{\sqrt{\rho^2/2+G(\mu)}+\sqrt{G(\mu)}})=4\sqrt{\mu}(\ln R-\ln r_0),$$

which implies $G'(\mu)\neq 0$.

According to the definition of $\Gamma(\mu)$,we have

$$\Gamma(\mu)=\sqrt{R^2/2+G(\mu)}-\sqrt{\rho^2/2+G(\mu)}.$$

(34)

Hence, we have $\Gamma'(\mu)>0$ in the case of $0<\mu<\mu_I$.

Step 2 If $\mu_I<\mu<\mu_{II}$, Using lemma 2 and lemma 3 , it holds that

$$\mu r_0^2(\mu)e^{-M^4(\mu)}>\frac{1}{2} \text{ and } L_1(\mu)<1/2,$$

which implies $G(\mu)<0$.

Integrating both sides of (30) over (ρ,r_0) and (r_0,R) respectively,we have

$$\frac{1}{\sqrt{-\mu G(\mu)}}(\arctan\frac{1}{2\sqrt{-\mu G(\mu)}}-\arctan\frac{1-2L_1(\mu)}{2\sqrt{-\mu G(\mu)}})=2(\ln r_0-\ln\rho)$$

(35)

and

$$\frac{1}{\sqrt{-\mu G(\mu)}}(\arctan\frac{1-2L_2(\mu)}{2\sqrt{-\mu G(\mu)}}-\arctan\frac{1}{2\sqrt{-\mu G(\mu)}})=2(\ln R-\ln r_0) .$$

(36)

Combining (33)to(36),we obtain

$$\frac{1}{\sqrt{-G(\mu)}}(\arctan\frac{\sqrt{R^2/2+G(\mu)}}{\sqrt{-G(\mu)}}-\arctan\frac{\sqrt{\rho^2/2+G(\mu)}}{\sqrt{-G(\mu)}})=2\sqrt{\mu}(\ln R-\ln r_0)$$

Which implies $G'(\mu)\neq 0,\ \mu_I<\mu<\mu_{II}$,Thus

$$\Gamma'(\mu)=G'(\mu)(\frac{1}{2\sqrt{R^2/2+G(\mu)}}-\frac{1}{2\sqrt{\rho^2/2+G(\mu)}})>0.$$

Step 3 If $\mu>\mu_{II}$, Using lemma 2 and lemma 3 , it holds that

$$\mu r_0^2(\mu)e^{-M^4(\mu)} > \frac{1}{2} \text{ and } L_1(\mu) < 1/2,$$

which implies $G(\mu) < 0$. Integrating both sides of (30) over (ρ, r_0) and (r_0, R) respectively,we also obtain (35) and (36). Combining (12)to(13),we obtain

$$\begin{cases} 1/2 - L_1(\mu_I) = -\sqrt{\frac{1}{2}\mu\rho^2 + \mu G(\mu)}, \\ 1/2 - L_2(\mu_I) = \sqrt{\frac{1}{2}\mu\rho^2 + \mu G(\mu)}. \end{cases}$$

(37)

Combining (35)to(37),we obtain

$$\frac{1}{\sqrt{-G(\mu)}}(\arctan\frac{\sqrt{R^2/2 + G(\mu)}}{\sqrt{-G(\mu)}} + \arctan\frac{\sqrt{\rho^2/2 + G(\mu)}}{\sqrt{-G(\mu)}}) = 2\sqrt{\mu}(\ln R - \ln\rho),$$

which implies $G'(\mu) \neq 0$.

According to the definition of $\Gamma(\mu)$,we have

$$\Gamma(\mu) = \sqrt{R^2/2 + G(\mu)} + \sqrt{\rho^2/2 + G(\mu)}.$$

Hence, we have $\Gamma'(\mu) > 0$ in the case of $\mu > \mu_{II}$

We then complete the proof of Theorem 2.2.

4. Conclusions

In this paper,we consider the degenerate nonlocal parabolic equation

$$u_t - \nabla\cdot(u^3\nabla u) = \frac{\lambda\exp(-u^4)}{(\int_\Omega \exp(-u^4)dx)^2},$$

with homogeneous Dirichlet boundary condition, where $\lambda > 0$, $\Omega = \{x \in R^2 : 0 < \rho < |x| < R\}$.

We prove that in the case of $0 < \lambda < |\partial\Omega|^2/2$,the equation corresponds a unique steady-state solution.

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