The Grassmannian Manifold and Controllability of the Linear Time-Invariant Control Systems

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Abstract

The idea discussed here are mainly to develope some interesting relationship between the differential geometry of certain curves and the controllability of linear time-invariant (LTI) control systems without considering any matrix riccati equation.

The problem based on the basic concepts of controllability is considered here. The two point boundary value problem (TPBVP) is described here as a flow in the Grassmannian manifold. Then a simple solution to determine a control function in the Grassmannian manifold is presented that transfer the system states from initial to final values and satisfies the conditions that are equivalent to the controllability of the systems.

Keywords: *Linear system, Control function, controllability, Grassmannian manifold.*

1. Introduction

Consider a LTI control system in the form of

$$\dot{x}(t)=Ax(t)+Bu(t), \qquad (1)$$

for $x \in R^n$, $u \in R^m$ and *A, B* are constant matrices.
The problem of controllability of the LTI control system (1) is considered here as a state transfer problem (STP). Thus we determine a control function u(t) that transfer the system states from the initial to final values within specified time interval of t seconds.

The problem of controllability is discussed in [1], as the state transfer problem. The solution of STP is given in [1], by computing the state-transition matrix. In [2], concept of controllability is considered as a STP and proposed several methods for synthesizing a control function for steering the given initial state of the system to the origin. In [3], the solution to STP is based on relating the given system to a family of phase-variable canonical form systems and then by using the technique of two-point interpolation.

The idea about differential geometry is given in [4]-[6]. It relates with the differential relation that stitches pieces of curves or surfaces together. It relates with the curves, surfaces, the functions that define them and transformation between the coordinates that can be used to specifies them.

Here, the concepts of differential geometry uses as a tool for analysis of control systems. The system (1), is described here as a flow in the Grassmannian manifold. Then a control function u(t) have synthesized in the Grassmannian manifold that transfer the system states from initial to final values and satisfies the conditions that are equivalent to the controllability of the LTI control systems.

Main objective here is that the control engineers should not restrict himself to any one of the tool but should be familiar with as many as possible for analysis purpose. The interesting relationship between the differential geometry of certain curves and the controllability of LTI control systems have been discussed in [7], with matrix riccati equation. But we have developed the same concepts with different approach, without considering any matrix riccati equation.

2. Brief Review of Literature

Some basic facts about the Grassmannian manifold and certain group actions are as follows. For detail, the reader should refer the exposition given by Doolin and Martin [6].

2.1. Lie group and group action:

Since the idea of a group is purely abstract algebraic idea, the definition of a group should involve only a set of element and some algebraic relations between them. A group is a set of elements, like a set of matrices, any pair of which can operate together to given another element that is also in the same set. In addition to different operation of the group, it requires some conditions, that is each element of the group and its inverse should be in the same group, one element in the group should be as an identity element and finally, the operation should be associative. For example, the set of all $n \times n$ unity matrices, $U(n)$, forms a group with the usual matrix product as the operation. The group $U(n)$ is a manifold, the product of two unity matrices is a unitary matrix and this operation is a manifold map [6]. The group with all these properties are called lie group or continuous transformation group. Lie group is a group that is a manifold and whose group operation yields C^{∞} manifold function and is associative.

Group Action:

The action of a group refers to members of a group operating on nonmembers. To understand the action of a group on a manifold, we consider the Lie group-G as a closed sub-group of the group of $N \times N$ invertible matrices, $Gl(N)$.

Now, we shall state the following definition from [6].
Definition: The group G is said to act on the manifold, $G^p(V)$, if there exist an infinitely differentiable function $\tau : G \times G^p(V) \to G^p(V)$ with the following properties.
i) for all $m \in G^p(V)$, $\tau(e, m) = m$, where e is the identity element.
ii) for all $g, h \in G$ and $m \in G^p(V)$, τg, $\tau(h,m) = \tau(gh,m)$.
We say in this case that the group G acts on $G^p(V)$.

Let $\alpha(t)$ be a one parameter sub-group of G with the following properties:
$\alpha(t)\alpha(h) = \alpha(t + h)$, $\alpha(0) = 1$, the identity.
Using these properties we can show that
$\alpha^{-1}(t) = \alpha(-t)$ and

$$\dot{\alpha}(t) = A\,\alpha(t) \tag{2}$$

where $A = \lim_{h \to 0} [\alpha(h) - \alpha(0)]/h.$

The matrix A is called the infinitesimal generator of the sub-group α. If α and A are real numbers, equation (2) has the solution $\alpha(t)e^{At}\alpha(0) = e^{At}$, confirming the connection of above mentioned properties with exponential. The same form holds if A is matrix, generating a matrix representation of the subgroup. Clearly the solution of the differential equation (2) is,

$$\alpha(t) = \exp At. \tag{3}$$

2.2. The Grassmannian manifold:

The Grassmannian manifold is the set of all p-dimensional subspaces of an n-dimensional vector space V. It is denoted by $G^p(V)$. Every p-dimensional subspace is denoted by W_G given by the linear transformation of $(n-p) \times p$ matrix G.

Let X and W be one dimensional subspaces of two dimensional vector space V, such that V is their direct sum: $V = X \oplus W$. By the direct sum meant that X and W have no subspaces in common except (0,0) of V, which will be referred to as the set {0}. The situation can be visualized in Fig.1.

The point p in Fig.1 belongs to the subspace W_G. It is specified uniquely by $x + Gx$, with G a real number (a 1 x 1 matrix) and $x \in X$. Every point in W_G is specified similarly by some x and every point in the plane except W itself belongs to a W_G for some G.

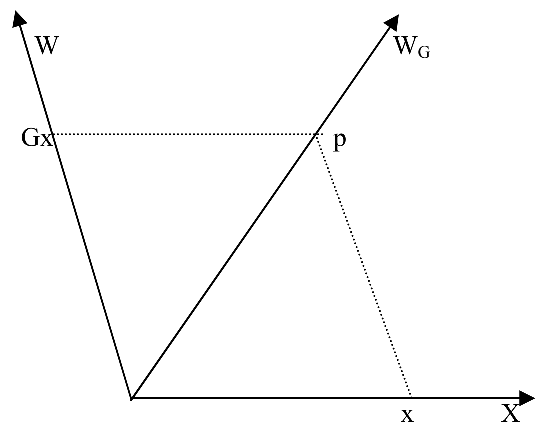

Fig. 1

The p-dim subspace X of V can be represented by a unique form, W_G in $G^p(V)$ as

$$W_G = \left[\begin{bmatrix} x \\ Gx \end{bmatrix} : x \in X \text{ and } G \in L(X, W) \right] \tag{4}$$

for some matrix G, if and only if $X_0 \cap W = \{0\}$. We leave out the proof of (4) and refer the reader to [5] & [6].

Let $Gl(N)$ be the set of all $N \times N$ invertible matrices. Also, let $\alpha(t)$ be a one-parameter sub-group of $Gl(N)$ and $G^p(V)$ be regarded as the Grassmannian manifold. The action of $\alpha(t)$ on $G^p(V)$ is given as :

$$Gl(N) \times G^p(V) \to G^p(V) \tag{5}$$

Therefore for $\alpha(t) \in Gl(N)$ and $W_G \in G^p(V)$, we have

$$(\alpha(t), W_G) \to \alpha(t)(W_G). \tag{6}$$

That meant an integral curve, $x(t)=\alpha(t)W_G$, in $G^p(V)$, is formed by the action of a one parameter sub-group $\alpha(t)$ of G as shown in Fig.2.

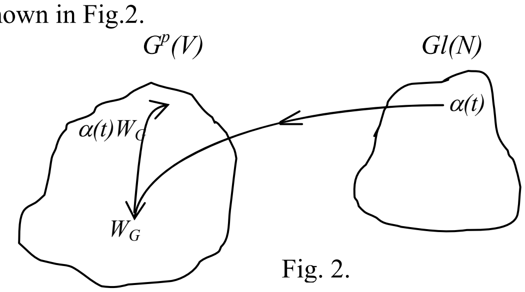

Fig. 2.

The mental picture is that the curve $x(t)$ being traced in $G^p(V)$ by the evolution on the continuous transformation of the initial p-dim. sub-space, W_G, under the action of the group.

3. Methodology

Let $G^p(V)$ be a Grassmannian manifold. A curve, $x(t)$, in a manifold $G^p(V)$, is defined to be a differentiable function,$x:R\rightarrow G^p(V)$,whose domain, (a, b), is an open interval of R. Then, $\dot{x}(t) = A\ (x(t))$, is a differential equation on $G^p(V)$. Now, a curve, $x(t)$, in $G^p(V)$ satisfying the equation $\dot{x}(t) = A\ (x(t))$, and such that $x(a) = x_a$. Such a curve is also called an integral curve starting from the point x_a.

At this stage, consider a curve $x(t)$, in $G^p(V)$, formed by the action of a one-parameter sub-group $\alpha(t)$ of G. That is, $x(t) = \alpha\ (t)x_a$. By differentiating this equation and by using the equation (2), we get the equation.

$$\dot{x}(t)=Ax(t) \tag{7}$$

Since A is the infinitesimal generator of a Lie group $\alpha(t)$, the determination of which corresponds to finding the solution of the differential equation (7).
This construction produces an ordinary differential equation in the form of the initial value problem with the initial value x_a. The vector field is generated by the matrix A and the flow, $\alpha(t) = expAt$ on the Grassmannian manifold $G^p(V)$.
Now, if W_G be the initial p-dim. subspace in the Grassmannian manifold, $G^p(V)$, then by the group action,

$$Gl(N) \times G^p(V) \rightarrow G^p(V). \tag{8}$$

Therefore for $W_G \in G^p(V)$ and $\alpha(t) \in Gl(N)$,

$$(\alpha(t),\ W_G) \rightarrow \alpha(t)\ (W_G). \tag{9}$$

If LTI control system (1) is in the form of

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{10}$$

for $x\in R^n$, $u\in R^m$, $t\in[0,T]$, with boundary conditions $x(0) = x_0$ and $x(T) = x_T$. Then with some efforts it is possible to describe the system (10), as a flow in the Grassmannian manifold,$G^p(V)$, as

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{11}$$

for $t\in[a,b]$, with boundary conditions $x_a=W_{GO}$ and $x_b=W_{GT}$.
A curve $x(t) = \alpha(t)W_G$ is an integral curve satisfying the equation (11) in $G^p(V)$, starting from the initial point $x_a=W_{GO.}$

For a necessary and sufficient condition for the existence of a unique solution of the boundary value problem,(11),if we consider $\alpha(t)$ in the matrix form, then from (4),we get,

$$\alpha(t)W_{G0} = \begin{bmatrix} \alpha_1(t)x_a \\ \alpha_3(t)x_a \end{bmatrix} \tag{12}$$

The eq^n.(12) can rewrite as

$$\alpha(t)W_{G0} = \begin{bmatrix} E(t) \\ \alpha_3(t)\ \alpha_1^{-1}(t)\ E(t) \end{bmatrix} \tag{13}$$

where $E(t) = \alpha_1(t)\ x_a \in X$. .Such a representation of $\alpha(t)$ is possible if and only if $\alpha^{-1}(t)$ exists. Hence the results. This is very important result, that gives condition for the existence of a unique solution of the TPBVP in the Grassmannian manifold, $G^p(V)$.

Now, with some efforts, a control function, $u(t)$, can be determine by relating the system (11) to a family of scalar differential equation and solving the problem latter by two-point interpolation in the Grassmannian manifold, $G^p(V)$, that affects a possible state transfer of the LTI systems and satisfies conditions that are equivalent to the controllability of the system.

4. Results

If system (10) is in the form of

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \tag{14}$$

for $x(t)\in R^n$, $u(t)\in R^m$ and $t = T = 1$sec., subject to $x_0 = [1\ 1\ -2]^T$, $x_T = [0\ 0\ 0]^T$. Then it is possible to define (14) as a flow in the Grassmannian manifold in the form of (11) as

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{15}$$

for $t\in[a,b]$, with boundary conditions $x_a=W_{GO}$ and $x_b=W_{GT}$.

A curve $x(t)$ be a solution curve satisfying the equation (15). For system to be controllable, it is necessary and sufficient condition that the solution curve of (15), that is $x(t)$, should be spanned by the controllability space formed by the n-vectors of the controllability matrix,Q_c,

$$Q_c=[B,\ AB,\ A^2B,\ \ldots\ldots\ldots,\ A^{n-1}B] \tag{16}$$

The controllability subspace $<A/B>$ is the space spanned by the columns of matrix B with respect to the linear transformation A, [2]. We do not assume that the pair (A, B) is controllable, of course when the pair (A,B) is not controllable, only some states are transferred to the origin. Therefore solution curve of (15),

$$x(t) = Q_c\alpha(t)W_G. \tag{17}$$

with initial and final values, from the boundary conditions, for the curve, $x(t)$, in the Grassmannian manifold, $x_a = W_{G0} = [15\ \ 7\ \ 1]^T$ and $x_b = W_{GT} = [0\ \ 0\ \ 0]^T$.

Now it is possible to determine a control function, $u(t)$, by relating the system (15) to a family of scalar differential equation and solving the problem latter by two-point interpolation in the Grassmannian manifold, $G^p(V)$, as

$$D^n x(t) + a_1 D^{n-1}x(t) + .. + a_{n-1}Dx(t) + a_n x(t) = u(t) \tag{18}$$

where, $a_1, a_2, \ldots, a_n$ are constants and operator 'D' represent differentiation with respect to time t. The system states can be described by equivalence between (15) and (18) as,

$$x(t) = x_1(t),\ Dx(t) = x_2(t),\ldots,\ D^{n-1}x(t) = x_n(t). \tag{19}$$

Here, instead of solving (18) for the response $x(t)$, we look it as a formula for a control function $u(t)$ in terms of the response $x(t)$ and look upon the desired state transfer of the LTI system as providing two point boundary conditions on an n-times differentiable function $x(t)$, in the Grassmannian manifold, $G^p(V)$. Thus by an interpolation technique a control function $u(t)$ can be synthesize as,

$$u(t) = -988 - 133t + 7267.5t^2 - 2111t^3 - 4437.5t^4 - 669t^5 \tag{20}$$

and the system states that satisfies the conditions that are equivalent to the controllability of the LTI system (15), in the Grassmannian manifold, $G^p(V)$,

$$x_1(t) = 15 + 7t + 0.5t^2 - 193.5t^3 + 282.5t^4 - 111.5t^5 \tag{21}$$

$$x_2(t) = 7 + t - 580.5t^2 + 1130t^3 - 557.5t^4 \tag{22}$$

$$x_3(t) = 1 - 1161t + 3390t^2 - 2230t^3. \tag{23}$$

The transfer characteristics of a control function, $u(t)$ and the system states $x_1(t)$, $x_2(t)$ and $x_3(t)$ are shown in Fig. 3, 4, 5 and 6 respectively.

5. Conclusion

Main objective here is that the control engineers should not restrict himself to any one of the tool but should be familiar with as many as possible for analysis purpose. The interesting relationship between the differential geometry of certain curves and the controllability of LTI control systems have developed here without considering any matrix riccati equations.

We have developed some interesting relationship between the curve obtained by the evolution of the continuous transformation of the initial condition under the action of the group in the Grassmannian manifold with the solution curve of TPBVP, that satisfies conditions that are equivalent to controllability of the LTI control systems.

This new idea of analysis of control systems using differential geometrical approach may help us greatly in near future. Graphical results shows possible state transfer of the linear time-invariant control system in the Grassmannian manifolds, by a control force, $u(t)$, that satisfies conditions that are equivalent to the controllability of the system.

This method has the flexibility of choosing the time interval $t = T$ sec. during which the transfer of the states from initial to final values are desired.

We can also extend the same idea for analysis of linear time-varying control system.

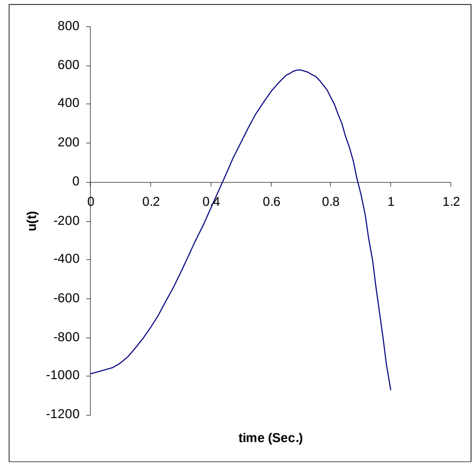

Fig.3 Transfer char. of a control function $u(t)$

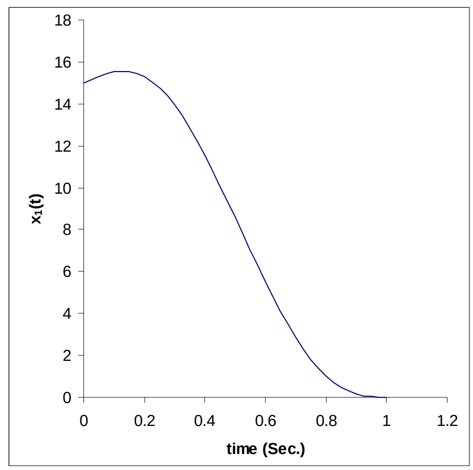

Fig. 4 Transfer char. of the state $x_1(t)$

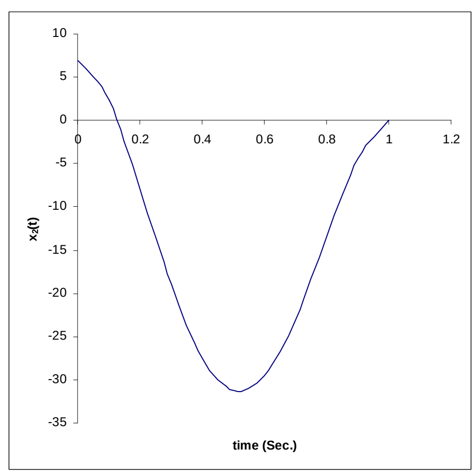

Fig. 5 Transfer char. of the state $x_2(t)$.

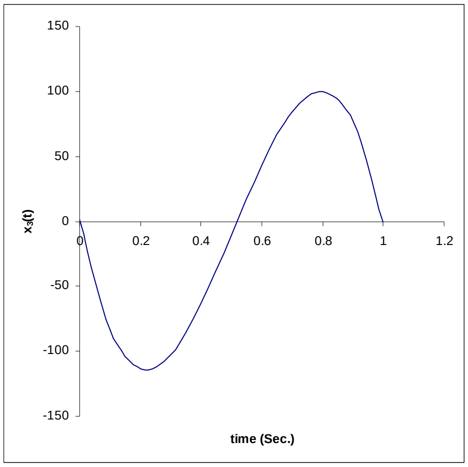

Fig. 6 Transfer char. of the state $x_3(t)$.

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